The Contact Yamabe Flow

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Zusammenfassung

Das Yamabe-Problem ist eine klassische Fragestellung aus der Differentialgeometrie. Es lautet: Ist eine gegebene kompakte und zusammenhängende Mannigfaltigkeit konform äquivalent zu einer Mannigfaltigkeit mit konstanter Skalarkrümmung? Diese Frage wurde 1960 von Yamabe formuliert. N. Trudinger and T. Aubin erzielten erste Resultate. Das Yamabe-Problem wurde von R. Schoen mit Hilfe des Positiven-Masse-Theorems im Jahr 1984 vollständig gelöst. Das CR-Yambabe Problem wurde 1987 von D. Jerison und J. M. Lee formuliert. Man fragt: Existiert für eine gegebene kompakte, strikt pseudokonvexe CR-Mannigfaltigkeit eine pseudohermitische Struktur, welche konstante Webster-Skalarkrümmung besitzt? Eine bejahende Lösung dieser Frage wurde von D. Jerison, J. M. Lee, N. Gamara und R. Yacoub im Jahr 2001 gefunden.

Das Yamabe-Problem wurde auch mit Hilfe von geometrischen Flüßen von R. Hamilton, R. Ye, H. Schwetlick, M. Struwe, und S. Brendle untersucht. Das Verhalten des Flußes ist auch für sich genommen interessant. In dieser Arbeit benutzen wir den sogenannten Kontakt-Yamabe-Flußum Lösungen des Kontakt-Yamabe-Problems zu finden. Das Kontakt-Yamabe-Problem ist eine natürliche Verallgemeinerung des CR-Yamabe-Problems.

Der Kontakt Yamabefluß ist eine degenerierte semilineare Wärmeleitungsgleichung. Wärmeleitungsgleichungen dieses Typs wurden bisher nicht weiter untersucht. Aus diesem Grund müssen wir zunächst einige geometrische und analytische Beobachtungen etablieren. Danach zeigen wir die Existenz einer Lösung für ein kleines Zeitintervall. Schließlich beweisen wir, daß der Yamabefluß für alle Zeiten existiert und gegen eine Lösung des Yamabe-Problems konvergiert, falls wir annehmen, daß entweder die Yamabe-Invariante negativ ist, oder das Anfangsdatum K-Kontakt ist.

Schlüsselwörter: Yamabe-Problem, Kontakt-Mannigfaltigkeit, Kontakt-Yamabefluß .

Abstract

The Yamabe problem is a classic problem in differential geometry concerning the question: whether a given compact and connected manifold is necessarily conformally equivalent to one of constant scalar curvature? It was formulated by Yamabe in 1960. Yamabe, N. Trudinger and T. Aubin made contribution to this problem, and it was completely solved by R. Schoen using positive mass thoerem in 1984. Later on D. Jerison and J. M. Lee introduced the CR Yamabe problem in 1987. That is, for a given compact, strictly pseudoconvex CR manifold, if it's possible to find a choice of pseudohermitian structure with constant Webster scalar curvature? This problem was solved in affirmative due to D. Jerison, J. M. Lee, N. Gamara and R. Yacoub in 2001.

A flow approach was also applied to the classic Yamabe problem by R. Hamilton, R. Ye, H. Schwetlick, M. Struwe and S. Brendle. The flow behavior has also its own interests. Here we use the contact Yamabe flow to find solutions of the contact Yamabe problem. The contact Yamabe problem is a natural generalization of the CR Yamabe problem.

The contact Yamabe flow corresponds to a degenerate semilinear heat equation. However the analytic theory regarding such heat equation has not been well studied up to now. For this reason we have to resort to some geometrical and also analytic observations. After we obtain the local existence in general, we prove the contact Yamabe flow exists for all time and tends to a solution of the contact Yamabe problem when the Yamabe invariant is negative or the initial data is K-contact.

Key words: Yamabe problem, contact manifold, contact Yamabe flow.

Contents

Chapter 1

Introduction

In this chapter let us briefly overview what we are going to do in this thesis. We will state our main theorems, explain how we organize this thesis and discuss some open questions. In particular, we assume that the reader is familiar with certain aspects in conformal geometry and in contact geometry. The analytic and geometric details of my thesis will be explained in the forthcoming chapters.

1.1 The main theorems

In this thesis we focus our interests on a Yamabe type flow on contact metric manifolds, i.e. we will use heat equations to solve Yamabe type problems on contact metric manifolds.

Let (M, θ_0, J, g_0) be a connected and compact contact metric manifold of dimension $2n+1$, where as usual θ_0, J denote the underlying contact form and the almost complex structure on the contact distribution given by $\ker(\theta_0)$. The Riemannian metric g_0 is associated with $d\theta_0$ and compatible with J (for details see chapter 3). The background contact form θ_0 defines a conformal class

$$[\theta_0] := \{\theta \in \Omega^1(M, \mathbb{R}) \,|\, \theta = f\theta_0, f > 0\}.$$

To each element θ in the conformal class $[\theta_0]$ one can assign a connection, called the generalized Tanaka connection (see [Tan89] and in addition section 3.5 in this thesis). The (generalized) Webster scalar curvature W then is the full trace of the curvature tensor associated to the Tanaka connection.

Any contact manifold (M, θ_0) admits a pair (J, g_0) consisting of an almost complex structure and an associated metric (not necessarily unique). With any such choice (J, g_0), (M, θ_0, J, g_0) is called a contact metric manifold. A Yamabe type problem on contact metric manifolds is to find a 1-form $\theta \in [\theta_0]$ such that the Webster scalar curvature w.r.t. θ is constant, i.e.

$$W(x) - \overline{W} = 0, \quad \forall\, x \in M, \tag{1.1}$$

where $\overline{W} := \dfrac{\int_M W(x,t)\theta \wedge d\theta^n}{\int_M \theta \wedge d\theta^n}$. To distinguish this Yamabe type problem on contact metric manifolds form the Riemannian Yamabe problem, we call it the contact Yamabe problem.

The semilinear, subelliptic equation (1.1) can be attacked also by considering the subparabolic analogue given by the *contact Yamabe flow*

$$\begin{cases} \frac{\partial\theta(x,t)}{\partial t} = (\overline{W}(t) - W(x,t))\theta(x,t) \\ \theta(x,0) = \theta^0 \in [\theta_0]. \end{cases} \tag{1.2}$$

As we will outline in chapters 4 and 5, the contact Yamabe problem on contact metric manifolds is a natural generalization of the CR Yamabe problem on CR manifolds that was initiated by D. Jerison and J. M. Lee in [JL87] and was completely solved by D. Jerison, J. M. Lee, N. Gamara and R. Yacoub (see [JL87], [JL89], [GY01] and [Gam01]).

In this thesis we use the flow approach to prove the following main theorems:

Theorem 1.1 *Let (M, θ_0, J, g_0) be a connected, compact contact metric manifold of dimension $2n + 1$.*

(a) The contact Yamabe flow (1.2) admits a smooth solution on a maximal time interval $[0, T), 0 < T \leq \infty$.

(b) If the contact Yamabe invariant $\lambda(M, [\theta_0])$ is negative, then there exists a contact metric structure $(M, \theta_\infty, J, g_\infty)$ with negative constant Webster scalar curvature. In particular, for any choice $\theta^0 \in [\theta_0]$ satisfying $W(\theta^0) < 0$ the solution $\theta(t)$ of (1.2) exists for all time and as $t \to \infty$ the Webster scalar curvature approaches some negative constant exponentially.

Theorem 1.2 *Let (M, θ_0, J, g_0) be a K-contact metric manifold. Then the contact Yamabe flow (1.2) with initial data $\theta^0 = \theta_0$ exists for all time and converges smoothly to a smooth limit θ_∞ with constant Webster scalar curvature.*

In Theorem 1.1(b) we can only prove $\theta(t)$ of the contact Yamabe flow (1.2) converges continuously to a limit $\theta(\infty)$, and the limit is actually smooth by (1.1). Better regularity can be proved in Theorem 1.2 where we prove all derivatives of the solution to the flow (1.2) are bounded uniformly in space and time which implies the smooth convergence. Theorem 1.2 finds a solution of the contact Yamabe problem on any K-contact metric manifold (for the definition see chapter 5). K-contact metric manifolds are a special class of contact metric manifolds. Any Sasakian manifold is a K-contact metric manifold and a K-contact metric manifold is not necessarily a CR manifold. For the definition of K-contact metric manifold see definition 5.10.

There exist great similarities between the contact Yamabe problem and the well-known Riemannian Yamabe problem in conformal geometry. To explain that in more details, let us briefly recall some basic facts concerning both problems. In the Riemannian Yamabe problem one considers the conformal class $[g_0]$ of a given Riemannian metric on a smooth manifold M. The problem is to find a new, conformally equivalent metric $g \in [g_0]$ such that its scalar curvature is constant.

For any conformally equivalent metric $g = u^{\frac{4}{n-2}} g_0$, the Riemannian Yamabe equation, i.e. the equation for constant scalar curvature of g becomes

$$-\frac{4(n-1)}{n-2} \triangle u + R_0 u = \lambda u^{\frac{n+2}{n-2}}, \tag{1.3}$$

where R_0 is the scalar curvature of g_0, Δ is the Laplace-Beltrami operator w.r.t. g_0 and λ is some constant. It was solved affirmatively by Yamabe, N. Trudinger, T. Aubin and in the remaining cases by R. Schoen using the positive mass theorem [Sch84].

This semi-linear elliptic problem has also been attacked by using a flow approach. The Riemannian Yamabe flow is defined by

$$\frac{\partial g}{\partial t} = (r_g - R_g) g, \tag{1.4}$$

where r_g is the average of the scalar curvature, i.e. $r_g = \frac{\int_M R_g dv_g}{\int_M dv_g}$. The Riemannian Yamabe flow can be reformulated as a heat equation for u:

$$\frac{\partial u}{\partial t} = \frac{4(n-1)}{n-2} u^{-\frac{4}{n-2}} \triangle u - R_0 u^{1-\frac{4}{n-2}} + r_g u. \tag{1.5}$$

By the works of Ye, Schwetlick, Struwe and Brendle (see [Ye94], [SS03] and [Bre05]), we know that the flow approach is effective too.

In contrast to the Riemannian Yamabe problem the conformal class on a contact metric manifold (M, θ_0, J, g_0) is now given by $[\theta_0]$. The role of the scalar curvature is now replaced by the Webster scalar curvature which is more suitable in the context of contact metric geometry (see more detailed discussion in chapters 3 and 4). If $\theta = u^{\frac{2}{n}} \theta_0$, then the contact Yamabe equation, i.e. the equation of constant Webster scalar curvature becomes

$$-2(2 + \frac{2}{n}) \triangle_P u + W_0 u = \lambda u^{\frac{n+2}{n}}, \tag{1.6}$$

where W_0 is the Webster scalar curvature of θ_0, $\triangle_P$ is the sublaplacian w.r.t. θ_0 (a degenerate second order elliptic operator, see chapter 3 for the exact definition) and λ is some constant.

As we have mentioned, this Yamabe problem was initiated by Jerison and Lee [JL87] on CR manifolds which are special contact metric manifolds. It was completely solved due to , [JL89], [GY01] and [Gam01]. The proof heavily relies on the analysis on CR manifolds, which goes back to [RS76].

Again, another approach to attack this problem is to use the corresponding flow, i.e. the contact Yamabe flow (1.2). According to (1.6) the contact Yamabe flow can be reformulated as

$$\frac{\partial u}{\partial t} = 2(2+\frac{2}{n})u^{-\frac{2}{n}}\triangle_P u - W_0 u^{1-\frac{2}{n}} + \overline{W}u. \tag{1.7}$$

It's a semi-linear sub-parabolic equation. Such kind of heat equation hasn't been well studied till now. Our approach to attack (1.7) is based on a combination of analytic and geometric techniques (see chapter 5). In particular, geometric observation play crucial roles in the proofs of Theorem 1.1(b) and 1.2.

1.2 Organization of the thesis

The structure of this thesis is as follows.

In the next chapter we give an overview of the standard Riemannian Yamabe problem, including the elliptic approach and the flow approach. It serves as the background materials which one can compare with the contact Yamabe problem. We make our effort to complete it with a rough introduction to the positive mass theorems. We focus on the powerful method of calculus of variations used in solving the Yamabe problem. Another interesting point we would like to introduce is that the Yamabe problem is not only a local problem but also a global one. J. M. Lee and T. H. Parker [LP87] give a very complete description of the Riemannian Yamabe problem. One can go there for more detailed materials.

In chapter 3 we introduce the contact geometry which is closely related to the contact Yamabe problem, especially the contact metric manifolds and CR manifolds. The main task of this chapter is to present the definition of the Webster scalar curvature. The Webster scalar curvature was first introduced by Tanaka and Webster independently on CR manifolds. This definition was then generalized by Tanno [Tan89] on contact metric manifolds. [Tan89] also studied conformal transformation of contact form. This is what we need to state the contact Yamabe problem on contact metric manifolds.

In chapter 4 we introduce the CR Yamabe problem on CR manifolds. Equation (1.6) comes in. It's a degenerate second order elliptic equation. We introduce the Folland-Stein Sobolev spaces and Folland-Stein Hölder spaces. They are the natural spaces made for the sub-elliptic operator $\triangle_P$. $\triangle_P$ satisfies the Hörmander condition. Some embedding theorems and a priori estimates are required to be introduced. We sketch the elliptic approach which is based on calculus of variations and blow-up analysis, due to Jerison and Lee ([JL87]).

In chapter 5 we define the contact Yamabe flow on contact metric manifolds. We prove it has local existence. This is first claimed in [CC02] for the unnormalized contact Yamabe flow on CR manifolds of dimension 3. We provide some basic properties of the contact Yamabe flow (1.2) and derive the evolution equation of the Webster scalar curvature.

In the second part of this chapter we assume the contact Yamabe invariant $\lambda(M, [\theta_0]) < 0$. Under such assumption we can choose some contact form $\theta^0 \in [\theta_0]$ with $W(\theta^0) < 0$, then we prove the contact Yamabe flow (1.2) has long-time existence and analyse its asymptotic behavior. The proof is based on the maximum principle. The C^0 norm of the solution is directly achieved. However, since a priori estimates for equation (1.7) are still lacking at this moment and the maximum principle is invalid at the first glance, we have to achieve the gradient estimate by resorting to some geometric information. The continuously convergence is also proved by using the maximum principle.

Finally we prove Theorem 1.2. It finds a solution of the contact Yamabe problem on any compact and connected K-contact metric manifold. The gradient estimates technique for the non-linear heat equation (1.7) works if we can control the derivatives in the direction of the Reeb vector field, so the basic property for the solution is crucial in our argument. It's a little bit surprising that we get uniform bounds for all derivatives of the solution. It then implies the global existence and convergence of the contact Yamabe flow (1.2).

1.3 Open questions and remarks

The Yamabe flow (1.4) has global existence. R. Hamilton, Ye, Schwetlick and Struwe (see [Ye94] and [SS03]) give different proofs. [SS03] treats only the more difficult part, it assumes the initial metric has positive scalar curvature. By Moser iteration it was shown the solution of (1.5) has C^0 bound on any finite time interval. The already known parabolic theory is enough to imply all bounds of its derivatives.

We have to ask the question if the contact Yamabe flow (1.2) has global existence. The proof of the global existence of the Riemannian Yamabe flow in [SS03] is adaptable to the contact Yamabe flow (1.2). Therefore on any finite time interval, the C^0 norm of u which is the solution of (1.7) can be achieved also. However, the sub-parabolic equation (1.7) has not been well studied. It's still unclear if the C^0 bound can imply the bounds of all higher derivatives. This is why the global existence problem is still under question.

After Hörmander's work [Hör67], a class of degenerate equations satisfying the Hörmander condition has drew much attention, such as [Hör67], [RS76], [CY92]. Most present works are devoted to its elliptic theory. However, the parabolic theory has not been studied sufficiently. We wonder if the parabolic theory of degenerate equations which satisfy the Hörmander condition is in fact as what one would expect, such as L^p estimates and Schauder estimates. If this is true, it would imply the convergence in Theorem 1.1(b) is smooth and the long-time existence of the contact Yamabe flow (1.2) in the case of $W(\theta_0) > 0$.

Chapter 2

The Riemannian Yamabe problem

In this chapter we give an introduction to the Riemannian Yamabe problem. It is divided into two parts, one is the elliptic approach and the other is the heat equation approach.

2.1 The elliptic approach

The Yamabe problem is a classic problem in differential geometry. There appeared various survey papers, see e.g. [SY94], [LP87] and references therein. J. M. Lee and T. H. Parker [LP87] give a detailed discussion of the the Yamabe problem along with a new argument unifying the work of T. Aubin [Aub76a] with that of R. Schoen [Sch84]. We would adopt their argument to explain the Yamabe problem in this chapter. For more detailed materials see [LP87].

2.1.1 History and motivations

A well known question in differential geometry is the question of whether a given compact and connected manifold is necessarily conformally equivalent to one of constant scalar curvature. This problem is known as the Yamabe problem because it was formulated by Yamabe [Yam60] in 1960. While Yamabe's paper claimed to solve the problem in the affirmative, it was found by N. Trudinger [Tru68] in 1968 that Yamabe's paper was seriously incorrect. Trudinger was able to correct Yamabe's proof in case the scalar curvature is non-positive. Aubin [Aub76a] improved Trudinger's result and the remaining cases were solved by Schoen [Sch84] by using positive mass theorem.

Let (M, g_0) be a Riemannian manifold, we say the metric g is conformal to g_0 if

$$g = ug_0,$$

for some smooth function $u > 0$. Let $[g_0]$ denote the conformal class of g_0, i.e. all the metrics conformal to g_0. We say (M, g_0) is conformally equivalent to one of constant scalar curvature means that one can find some metric g conformal to g_0 with respect to which the scalar curvature is constant.

Yamabe initiated his problem with the aim to construct Einstein metric. (M, g) is said to be Einstein if its Ricci curvature tensor is a constant multiple of the metric.

Another motivation to consider the Yamabe problem comes from conformal geometry itself. Riemannian differential geometry originated in attempts to generalize the highly successful theory of compact surfaces. From the earliest days, conformal changes of metric have played an important role in surface theory. For example, one consequence of the famous uniformization theorem of complex analysis is the fact that every surface has a conformal metric of constant (Gaussian) curvature. In higher dimension cases it is natural to seek a conformal change of metric that makes the scalar curvature constant. Thus we also are led to the Yamabe problem.

Given a compact Riemannian manifold (M, g) of dimension $n \geq 3$, find a metric conformal to g of constant scalar curvature.

We always assume M is connected in the thesis.

The Yamabe invariant $\lambda(M, g)$ is central to the analysis of the Yamabe problem (see (2.4) for the definition). The solution of the Yamabe problems can be summarized by three main theorems.

(a) ([Yam60], [Tru68] and [Aub76a]) Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. If the Yamabe invariant $\lambda(M, g)$ is strictly less than that of the standard n-sphere, the Yamabe problem can be solved on (M, g).

    This result shifts the focus of the proof from analysis to the problem of understanding the essentially geometric invariant $\lambda(M, g)$. The obvious approach to show that $\lambda(M, g) < \lambda(S^n)$ is to find the desired test function. T. Aubin [Aub76a] sought such a test function compactly supported in a small neighborhood of a point $P \in M$. By carefully studying the local geometry of M near P in normal coordinates, he was able to construct such test functions in many cases, proving the following theorem.

(b) ([Aub76a]) Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Then $\lambda(M, g) \leq \lambda(S^n)$. Furthermore if M has dimension $n \geq 6$ and not locally conformally flat then $\lambda(M, g) < \lambda(S^n)$.

    The remaining cases are more difficult because the local conformal geometry does not contain sufficient information to conclude that $\lambda(M, g) < \lambda(S^n)$. These cases require the construction of a global test function.

This was done by R. Schoen [Sch84] in 1984. His theorem completes the solution of the Yamabe problem.

(c) ([Sch84]) If M has dimension 3, 4, or 5, or if M is locally conformally flat, then $\lambda(M,g) < \lambda(S^n)$ unless M is conformal to the standard sphere.

Schoen's proof introduced two important new ideas. First, he recognized the key role of the Green function for the conformal Laplacian; his test function was the Green function with the singularity smoothed out. Second, he discovered the unexpected relevance of the positive mass theorem of general relativity. A curious feature of Schoen's proof is that it works only in the cases not covered by Aubin's theorem.

The solution of the Yamabe problem marks a milestone in the development of the theory of nonlinear partial differential equations. Semi-linear equations of the form (1.3) with critical exponent arise in many contexts and have been long studied by analysts. This is the first time that such a equation has been completely solved.

2.1.2 Basic materials

Yamabe attempted to solve the Yamabe problem using techniques of variations and elliptic partial equations.

Any metric conformal to g can be written as $\widetilde{g} = e^{2f}g$, where f is a smooth function on M. Let R and $\widetilde{R}$ denote the scalar curvatures of g and $\widetilde{g}$ respectively, a direct computation shows

$$\begin{aligned}\widetilde{R} &= \widetilde{g}^{ij}\widetilde{R}_{ij}\\ &= e^{-2f}[R - 2(n-1)\triangle f - (n-2)(n-1)|\nabla f|^2].\end{aligned}$$

Take $u = e^{\frac{n-2}{2}f}$, i.e. $\widetilde{g} = u^{\frac{4}{n-2}}g$, we can simplify the transformation law as

$$\widetilde{R} = (-\frac{4(n-1)}{n-2}\triangle u + Ru)u^{-\frac{n+2}{n-2}}.$$

The Laplace-Beltrami operator is w.r.t. the metric g. For simplicity we denote $a = \frac{4(n-1)}{n-2}$ and $p = \frac{2n}{n-2}$ throughout this chapter. Thus for $\widetilde{g} = u^{p-2}g$,

$$\widetilde{R} = (-a\triangle u + Ru)u^{1-p}. \tag{2.1}$$

Therefore the Yamabe problem is to solve the Yamabe equation :

$$-a\triangle u + Ru = \lambda u^{p-1} \tag{2.2}$$

for some constant λ. If $u > 0$ is a smooth solution of the Yamabe equation (2.2), then for $\widetilde{g} = u^{p-2}g$, $\widetilde{R} = \lambda$.

Denote $\Box$ as $-a\triangle + R$, called the conformal Laplacian. The Yamabe problem is a sort of "nonlinear eigenvalue problem". The analytic properties of the equation $\Box u = \lambda u^q$ depend critically on the value of the exponent q: when $q = 1$, the equation is just the linear eigenvalue problem for $\Box$. When q is close to 1, its analytic behavior is quite similar to that of the linear case, and the problem is easily solved. When q is very large, the methods based on linear theory break down altogether. It happens that the exponent $q = p - 1$ that occurs in the Yamabe equation is precisely the critical value, below which the equation is easy to solve and above which it may be impossible. This accounts for the analytic complexity of the Yamabe Problem.

Yamabe observed that the Yamabe equation (2.2) is the Euler-Lagrange equation of the Yamabe functional defined by

$$Y(\widetilde{g}) = \frac{\int_M \widetilde{R} dv_{\widetilde{g}}}{(\int_M dv_{\widetilde{g}})^{\frac{n-2}{n}}},$$

where $\widetilde{g}$ is allowed to vary over metrics conformally to g and $dv_{\widetilde{g}}$ is the volume form w.r.t. $\widetilde{g}$.

In fact we can say more about the Yamabe functional

$$Y(g) = \frac{\int_M R_g dv_g}{(\int_M dv_g)^{2/p}}.$$

Let $\mathcal{M}$ denote the space of all metrics, then for $g \in \mathcal{M}$ and $h \in T_g\mathcal{M}$ the first variation formula is

$$DY_g(h) = \frac{1}{V^{\frac{n-2}{n}}} \int_M < -R_{ij} + \frac{1}{2} R_g g_{ij} - \frac{n-2}{2n} r_g g_{ij}, h_{ij} >_g dv_g, \qquad (2.3)$$

where V is the volume, r_g is the average scalar curvature, $< \cdot, \cdot >_g$ is the inner product with respect to the metric g (for the computation one can see [Sch89]).

From the first variation formula (2.3), one can see the following well known conclusions.

Proposition 2.1 *(a) If g is a critical point of the Yamabe functional Y in the conformal class $[g]$, then*

$$R_g = r_g,$$

i.e. g has constant scalar curvature.

(b) If g is a critical point of the Yamabe functional Y in $\mathcal{M}$, then

$$R_{ij} = \frac{1}{n} r g_{ij},$$

i.e. g is an Einstein metric.

In fact, if g is a critical point of Y, then $DY_g(h) = 0$ for any $h \in T_g\mathcal{M}$. In particular, for the conformally variations h, $DY_g(h) = 0$ which implies $R_g = r_g$. Therefore, by the first variation formula again, we have $R_{ij} = \frac{r_g}{n} g_{ij}$.

(c) *If $R_g = r_g$, then g is a critical point of Y in the conformal class $[g]$.*

(d) *Einstein metrics are critical points of the Yamabe functional Y.*

■

To construct Einstein metrics, the first step is to minimize the Yamabe functional Y in every conformal class, then to maximize Y among all conformal classes. In view of (a) and (c), the first step is equivalent to find metric of constant scalar curvature in its conformal class. This is the motivation that Yamabe proposed the Yamabe problem.

From the above argument, one can try to find a solution of the Yamabe problem by using calculus of variations. For $\widetilde{g} = u^{p-2}g$,

$$Y(\widetilde{g}) = \frac{\int_M \widetilde{R} dv_{\widetilde{g}}}{(\int_M dv_{\widetilde{g}})^{2/p}} = \frac{\int_M (a|\nabla u|_g^2 + R_g u^2) dv_g}{(\int_M u^p dv_g)^{2/p}}.$$

So for a given Riemannian manifold (M, g), it's natural to define the following constrained extremal problem:

$$\lambda(M, g) = \inf\{A(u) | B(u) = 1, u \in W^{1,2}(M, g)\}, \tag{2.4}$$

where

$$A(u) = \int_M (a|\nabla u|_g^2 + R_g u^2) dv_g$$

and

$$B(u) = \int_M |u|^p dv_g.$$

Therefore $\lambda(g) = \inf\{Y(\widetilde{g}) : \widetilde{g} \in [g]\}$.

The conformal Laplacian $\Box$ is conformally invariant in the following sense. If $\widetilde{g} = \varphi^{\frac{4}{n-2}} g$ is a metric conformal to g and $\widetilde{\Box}$ denotes the conformal Laplacian with respect to $\widetilde{g}$, then

$$\widetilde{\Box}(\varphi^{-1}u) = \varphi^{1-p}\Box u.$$

Thus

$$Y((\varphi^{-1}u)^{p-2}\widetilde{g}) = \frac{\int_M \widetilde{\Box}(\varphi^{-1}u)(\varphi^{-1}u) dv_{\widetilde{g}}}{\int_M |\varphi^{-1}u|^p dv_{\widetilde{g}}} = \frac{\int_M \Box u u dv_g}{\int_M |u|^p dv_g} = Y(u^{p-2}g).$$

Therefore the constant $\lambda(M, g)$ is an invariant of the conformal class $[g]$, called the Yamabe invariant . Its value is central to the analysis of the Yamabe problem.

2.1.3 The solution when $\lambda(M,g) < \lambda(S^n,\overline{g})$

In this section we use calculus of variations to prove that the Yamabe problem can be solved on a general compact Riemannian manifold (M^n, g) provided that its Yamabe invariant $\lambda(M,g) < \lambda(S^n,\overline{g})$, where $\overline{g}$ is the standard metric on the sphere. This is due to Yamabe, Trudinger and Aubin. Trudinger's [Tru68] modification of Yamabe's proof works whenever $\lambda(M,g) \leq 0$. In fact, he shows that there is a positive constant Λ such that the proof works when $\lambda(M,g) < \Lambda$. In 1976, Aubin [Aub76a] extended Trudinger's result by showing, in effect, that $\Lambda = \lambda(S^n,\overline{g})$. This established:

Theorem 2.2 *([Yam60], [Tru68] and [Aub76a]) Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. If $\lambda(M,g) < \lambda(S^n,\overline{g})$, then the extremal problem $\lambda(M,g)$ in (2.4) is attained by a positive, smooth solution of the Yamabe equation (2.2). Thus, the metric $\widetilde{g} = u^{\frac{4}{n-2}} g$ has constant scalar curvature $\widetilde{R} = \lambda(M,g)$.*

This theorem is proved by using calculus of variations. Take a minimizing sequence $u_i \in W^{1,2}(M,g)$, satisfying $B(u_i) = 1$, to minimize functional A. Since the sequence $||u_i||_{W^{1,2}(M,g)}$ is bounded, there exists a subsequence (still denoted as u_i) converges weakly to u in $W^{1,2}(M,g)$. Then $A(u) = \lambda(M,g)$. But $B(u) \neq 1$ may happen because the embedding $W^{1,2}(M,g) \hookrightarrow L^p$ is not compact. In particular, the limit function u may be identically zero.

So we consider a perturbed extremal problem for $2 \leq q < p$.

$$\lambda_q(g) = \inf\{A(u)|B(u) = 1, u \in W^{1,2}(M,g)\},$$

where

$$A(u) = \int_M (a|\nabla u|_g^2 + R_g u^2) dv_g$$

and

$$B(u) = \int_M |u|^q dv_g.$$

Since $W^{1,2}(M,g) \hookrightarrow L^q$ is compact for $2 \leq q < p$, by the same argument one can get a limit $u_q \in W^{1,2}(M,g)$ for any $2 \leq q < p$ attained the perturbed extremal problem. Therefore, u_q is a weak solution of

$$a\triangle u_q - Ru_q + \lambda_q u_q^{q-1} = 0.$$

Replacing u_q by $|u_q|$, one can assume u_q is nonnegative. A regularity result of N. Trudinger [Tru68] is the following.

Lemma 2.3 *If $u_q \in W^{1,2}(M,g)$ is a nonnegative and nonzero weak solution of the equation*

$$a\triangle u_q - Ru_q + \lambda_q u_q^{q-1} = 0,$$

for $2 \leq q \leq p$, then u_q is smooth and $||u_q||_{C^{2,\beta}} \leq C$, where C is a constant depends only on (M,g) and $0 < \beta < 1$. The strong maximum principle implies that $u_q > 0$.

When $2 \leq q < p$, iteration of the standard L^p estimates for the Laplace-Beltrami operator and the L^p version Sobolev embedding theorems show that u_q is smooth. For the critical case $q = p$, N. Trudinger [Tru68] chose suitable test functions and used Moser iteration to get u_p is smooth.

Thus u_q is a positive, smooth solution of

$$a\triangle u_q - Ru_q + \lambda_q u_q^{q-1} = 0$$

for $2 \leq q < p$. The distribution form of it is

$$\int_M (a\nabla\eta \cdot \nabla u_q + Ru_q\eta - \lambda_q u_q^{q-1}\eta) dv_g = 0$$

for any $\eta \in C^\infty(M)$.

Since $\{||u_q||_{W^{1,2}(M,g)}\}$ are bounded, there exists a subsequence $u_q \rightharpoonup u$ in $W^{1,2}(M,g)$ as $q \to p$ such that u satisfies

$$\int_M (a\nabla\eta \cdot \nabla u + Ru\eta - \lim_{q\to p} \lambda_q u^{p-1}\eta) dv_g = 0,$$

for any $\eta \in C^\infty(M)$.

Studying the behavior of λ_q when q tends to p becomes necessary now. This is what the following lemma concerns.

Lemma 2.4 *([Aub76a]) If $\int_M dv_g = 1$, then $|\lambda_q|$ is nonincreasing as a function of $q \in [2,p]$; and if $\lambda(M,g) \geq 0$, λ_q is continuous from the left, therefore there exists a subsequence λ_q which tends to $\lambda(M,g)$.*

The only remaining thing needded to be shown is that u is nonzero, and this is where the condition $\lambda(M,g) < \lambda(S^n, \overline{g})$ enters. Aubin [Aub76a] observed that the best constant in the Sobolev inequality is the same for all compact manifolds in the following sense. For any M and any $\epsilon > 0$, there exists $C_{M,\epsilon}$ such that

$$(\lambda(S^n) - \epsilon)(\int_M |f|^p dv_g)^{\frac{2}{p}} \leq a \int_M |df|^2 dv_g + C_{M,\epsilon} \int_M |f|^2 dv_g \tag{2.5}$$

for all $f \in W^{1,2}(M)$. Inequality (2.5) is proved by transferring the inequality from Euclidean space to manifolds via Riemannian normal coordinates and a partition of unity.

For each u_q, q sufficiently close to p, applying lemma 2.4 and (2.5) it follows $||u_q||_{L^2} \geq \delta$ for some $\delta > 0$, actually one can choose sufficiently small ϵ such that

$$\begin{aligned}
& a\int_M |du_q|^2 dv_g + C_{M,\epsilon}\int_M |u_q|^2 dv_g \\
\geq\ & (\lambda(S^n,\overline{g})-\epsilon)(\int_M |u_q|^p dv_g)^{2/p} \\
\geq\ & (\lambda_q+\delta)(\int_M |u_q|^p dv_g)^{2/p} \\
\geq\ & (\lambda_q+\delta)(\int_M |u_q|^q dv_g)^{2/q} \\
=\ & \lambda_q+\delta \\
=\ & a\int_M |du_q|^2 dv_g + \int_M R_g|u_q|^2 dv_g + \delta.
\end{aligned}$$

Since $W^{1,2}$ is compactly embedded in L^2, the same bound holds on u, and hence u is nonzero. Then by Lemma 2.3, u is a smooth and positive solution of the Yamabe equation (2.2). Thus u is a solution of the Yamabe equation (2.2). We have finished the sketch of the proof of Theorem 2.2.

An alternative proof of Theorem 2.2 has been given by Uhlenbeck , which does not require the result that the Sobolev constant is independent of M as (2.5). Instead, assuming u_q doesn't converge, she used Riemannian normal coordinates to transport u_q to R^n in such a way that the transplanted functions converge in $C^1(R^n)$. The limit function $\widetilde{u}$ then is shown to contradict Sobolev's inequality on R^n if $\lambda(M,g) < \lambda(S^n,\overline{g})$. This kind of blow-up analysis was first introduced by Sacks and Uhlenbeck in [SU81] and of great importance in nonlinear problems.

2.1.4 The solutions on the standard sphere

By Theorem 2.2, the standard sphere plays a special role in the Yamabe problem. In this section we study the conformal geometry of the standard sphere.

Let $P=(0,...,0,1)$ be the north pole on $S^n\subset R^{n+1}$. Stereographic projection $\sigma: S^n - P\to R^n$ is defined by $\sigma(\zeta^1,...,\zeta^n,\xi)=(x^1,...,x^n)$ for $(\zeta,\xi)\in S^n-P$, where

$$x^j=\zeta^j/(1-\xi).$$

σ is a conformal diffeomorphism. In fact, if ds^2 is the Euclidean metric on R^n, then

$$\rho^*\overline{g}=4(1+|x|^2)^{-2}ds^2,$$

where ρ denotes σ^{-1} and $\overline{g}$ is the standard metric on S^n with constant sectional curvature 1. This can be written as

$$\rho^*\overline{g}=4u_1^{\frac{4}{n-2}}ds^2,$$

with $u_1(x) = (1 + |x|^2)^{(2-n)/2}$.

By means of stereographic projection, it is simple to write down conformal diffeomorphisms of the sphere. The group of such diffeomorphisms is generated by the rotations, together with maps of form $\sigma^{-1}\tau_v\sigma$ and $\sigma^{-1}\delta_\alpha\sigma$, where $\tau_v, \delta_\alpha : R^n \to R^n$ are respectively translation by $v \in R^n$:

$$\tau_v(x) = x - v,$$

and dilation by $\alpha > 0$:

$$\delta_\alpha(x) = \alpha^{-1}x.$$

The spherical metric on R^n transforms under dilation to

$$\delta_\alpha^* \rho^* \overline{g} = 4u_\alpha^{4/(n-2)} ds^2, \tag{2.6}$$

where $u_\alpha(x) = (\frac{|x|^2+\alpha^2}{\alpha})^{(2-n)/2}$. We can compute that

$$-a\triangle u_\alpha = 4n(n-1)u_\alpha^{p-1}, \tag{2.7}$$

therefore $R_{\delta_\alpha^* \rho^* \overline{g}} = n(n-1)$.

u_α will be important test functions used later.

It is an important fact that the metric $\overline{g}$ minimizes the Yamabe functional Y in the conformal class $[\overline{g}]$. This result is due originally to Aubin [Aub76b], and independently to G.Talenti [Tal76].

J. M. Lee and T. H. Parker [LP87] give a simpler way to show this, by combing the results of K. Uhlenbeck and M. Obata. First by means of a "renormalization" approach, due to Uhlenbeck, there exists a positive smooth function φ on S^n satisfying $Y(\varphi^{p-2}\overline{g}) = \lambda(S^n, \overline{g})$.

It can also be shown, in fact, using methods of H. Brezis, L. Nirenberg, E. Lieb and P.-L. Lions (see [BL83], [BN83] and [Lio83]), that any minimizing sequence on the sphere can be renormalized to converge to a smooth extremal. Second, thanks to the following

Proposition 2.5 *([Oba72]) If g is a metric on S^n that is conformal to the standard metric $\overline{g}$ and has constant scalar curvature, then up to a constant scale factor, g is obtained from $\overline{g}$ by a conformal diffeomorphism of the sphere.*

In fact it is shown that such metric g is Einstein. Considering the given metric g as "background" metric on the sphere, we can write $\overline{g} = \varphi^{-2}g$, where $\varphi \in C^\infty(S^n)$ is strictly positive. One can compute

$$\overline{R}_{jk} = R_{jk} + \varphi^{-1}[(n-2)\varphi_{,jk} - (n-1)\varphi^{-1}|\nabla\varphi|^2 g_{jk} + \triangle\varphi g_{jk}],$$

in which the covariant derivatives and Laplacian are taken with respect to g. If $B_{jk} = R_{jk} - (R/n)g_{jk}$ represents the traceless Ricci tensor, then since $\overline{g}$ is Einstein,

$$0 = \overline{B}_{jk} = B_{jk} + (n-2)\varphi^{-1}(\varphi_{,jk} - \frac{1}{n}\triangle\varphi g_{jk}).$$

Since the scalar curvature R is constant, the contracted Bianchi identity implies that the divergence $R^i_{m,i}$ of the Ricci tensor vanishes identically, and thus so also does $B^i_{m,i}$. Because B_{jk} is traceless, integration by parts gives

$$\begin{aligned}\int_{S^n} \varphi|B|^2 dv_g &= \int_{S^n} \varphi B_{jk} B^{jk} dv_g \\ &= -(n-2)\int_{S^n} B^{jk}(\varphi_{,jk} - \frac{1}{n}\triangle\varphi g_{jk})dv_g \\ &= -(n-2)\int_{S^n} B^{jk}\varphi_{,jk} dv_g \\ &= -(n-2)\int_{S^n} B^{jk}_{,k}\varphi_j dv_g = 0.\end{aligned}$$

Thus B_{jk} must be identically zero, and so g is Einstein.

On the other hand, since g is conformal to the standard metric $\overline{g}$ on the sphere, which is locally conformally flat. These imply that g has constant curvature, and so (S^n, g) is isometric to a standard sphere. The isometry is the desired conformal diffeomorphism. It establishes:

Theorem 2.6 *The Yamabe functional Y on $(S^n, \overline{g})$ is minimized by constant multiplies of the standard metric and its images under conformal diffeomorphisms. These are the only metrics conformal to the standard one on S^n that have constant scalar curvature.*

2.1.5 Aubin's results

In this section we introduce two theorems, due to Aubin [Aub76a], concerning the Yamabe invariant $\lambda(M, g)$.

For any (M, g), from the definition of $\lambda(M, g)$ we see that $\lambda(M, g)$ is bounded from below and above. In fact there exists an uniform upper bound.

Theorem 2.7 *([Aub76a])Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Then $\lambda(M, g) \leq \lambda(S^n, \overline{g}) = 4n(n-1)||u_\alpha||_p^{p-2}$, where $(S^n, \overline{g})$ is the n-sphere with the standard metric.*

This fact is based on a local argument. Since $u_\alpha = (\frac{|x|^2+\alpha^2}{\alpha})^{(2-n)/2}$ satisfies (2.7), it follows that $\lambda(S^n, \overline{g}) = 4n(n-1)||u_\alpha||_p^{p-2}$. For any fixed $\epsilon > 0$, let B_ε denote the ball of radius ϵ in R^n, and choose a smooth radial cutoff function

$0 \leq \eta \leq 1$ supported in $B_{2\epsilon}$, with $\eta = 1$ on B_ϵ. Consider the smooth, compactly supported function $\varphi = \eta u_\alpha$. Since φ is a function of $r = |x|$ alone,

$$\begin{aligned} \int_{R^n} a|\nabla\varphi|^2 dx &= \int_{B_{2\epsilon}} (a\eta^2|\nabla u_\alpha|^2 + 2a\eta u_\alpha < \nabla\eta, \nabla u_\alpha > + a u_\alpha^2 |\nabla\eta|^2) dx \\ &\leq \int_{R^n} a|\partial_r u_\alpha|^2 dx + C\int_{A_\epsilon} (u_\alpha|\partial_r u_\alpha| + u_\alpha^2) dx, \end{aligned}$$

where A_ϵ denotes the annulus $B_{2\epsilon} - B_\epsilon$. To estimate these terms we observe that

$$\partial_r u_\alpha = (2-n) r\alpha^{-1} (\frac{r^2+\alpha^2}{\alpha})^{-n/2},$$

so $u_\alpha \leq \alpha^{(n-2)/2} r^{2-n}$ and $|\partial_r u_\alpha| \leq (n-2)\alpha^{(n-2)/2} r^{1-n}$. Thus, for fixed ϵ, the second term is $O(\alpha^{n-2})$ as $\alpha \to 0$. For the first term,

$$\begin{aligned} \int_{R^n} a|\partial_r u_\alpha|^2 dx &= \lambda(S^n, \overline{g})(\int_{B_\epsilon} u_\alpha^p dx + \int_{R^n - B_\epsilon} u_\alpha^p dx)^{\frac{2}{p}} \\ &\leq \lambda(S^n, \overline{g})(\int_{B_{2\epsilon}} \varphi^p dx + \int_{R^n - B_\epsilon} \alpha^n r^{-2n} dx)^{2/p} \\ &= \lambda(S^n, \overline{g})(\int_{B_{2\epsilon}} \varphi^p dx)^{2/p} + O(\alpha^{n-2}). \end{aligned}$$

Thus we have constructed a test function $\varphi = \eta u_\alpha$ and shown that

$$\lambda(R^n, ds^2) \leq \lambda(S^n, \overline{g}) + C\alpha^{n-2}.$$

Let $\alpha \to 0$, we see that the Sobolev quotient of φ on R^n is less than $\lambda(S^n, \overline{g})/a$. The fact $\lambda(M, g) \leq \lambda(S^n, \overline{g})$ can be proved in a similar way, for more details see [LP87].

From Theorem 2.2 and 2.7, the Yamabe problem seems to be almost finished except the cases $\lambda(M, g) = \lambda(S^n, \overline{g})$. They shift the focus of the proof from analysis to the problem of understanding the essentially geometric invariant $\lambda(M, g)$. One main step is the following.

Theorem 2.8 *([Aub76a]) If M has dimension $n \geq 6$ and is not locally conformally flat then $\lambda(M, g) < \lambda(S^n, \overline{g})$.*

The obvious approach to show $\lambda(M, g) < \lambda(S^n, \overline{g})$ is to find a "test function" φ with $Y(\varphi^{p-2} g) < \lambda(S^n, \overline{g})$. Aubin sought such a test function compactly supported in a small neighborhood of a point $P \in M$. By carefully studying the local geometry of M near P in normal coordinates, he was able to construct such test function in the above case.

There exists normal coordinates for some metric g within the conformal class. The freedom in choice of g will enable us to find coordinate systems that simplify the local geometry. A well known and convenient coordinates is the conformal normal coordinates. J. M. Lee and T. H. Parker [LP87] use it to unify the results of Aubin and Schoen.

Conformal normal coordinates. Let (M, g_0) be a Riemannian manifold and $P \in M$. For each $N \geq 2$ there exists a metric g conformal to g_0 such that

$$det(g_{ij}) = 1 + O(r^N),$$

where $r = |x|$ in g-normal coordinates at P.

In conformal normal coordinates, if $N \geq 5$, the scalar curvature of g satisfies $R = O(r^2)$ and $\triangle R = -\frac{1}{6}|W|^2$ at P. The Weyl tensor W is given by

$$\begin{aligned} W_{ijkl} &= R_{ijkl} - \frac{1}{n-2}(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}) \\ &\quad + \frac{1}{(n-1)(n-2)}R(g_{ik}g_{jl} - g_{il}g_{jk}). \end{aligned}$$

If the dimension $n = 3$, W vanishes identically; For $n \geq 4$ the Weyl tensor vanishes identically if and only if M is locally conformally flat. The existence of such conformal normal coordinates can be proved by using Graham's work and the following formula.

In g-normal coordinates,

$$\begin{aligned} det(g_{ij}) &= 1 - \frac{1}{3}R_{ij}x^i x^j - \frac{1}{6}R_{ij,k}x^i x^j x^k \\ &\quad -(\frac{1}{20}R_{ij,kl} + \frac{1}{90}R_{pijm}R_{pklm} - \frac{1}{18}R_{ij}R_{kl})x^i x^j x^k x^l + O(r^5), \end{aligned}$$

where the curvatures are evaluated at P.

Using the conformal normal coordinates, Theorem 2.8 can be proved now. It's similar to the argument of Theorem 2.7.

Let $\{x^i\}$ be the conformal normal coordinates on a neighborhood of $P \in M$. Recalling the notation of Theorem 2.7, let $\varphi = \eta u_\alpha$ in x-coordinates, where η is a cutoff function supported in $B_{2\epsilon}$. Since $det(g_{ij}) = 1 + O(r^N)$ in conformal normal coordinates, in a similar way as in Theorem 2.7 one can establish:

$$A(\varphi) = \int_{B_{2\epsilon}} a|\nabla\varphi|^2 + R\varphi^2 dv_g \leq \lambda(S^n)||\varphi||_p^2 + C\alpha^{n-2} + \int_{B_{2\epsilon}} R\varphi^2 dx.$$

We already know in conformal normal coordinates $\{x^i\}$, $R = O(r^2)$ and $\triangle R(P) = -\frac{1}{6}|W(P)|^2$, so

$$\begin{aligned} \int_{B_{2\epsilon}} R\varphi^2 dx &\leq \int_{B_\epsilon} Ru_\alpha^2 dx + C\int_{A_\epsilon} u_\alpha^2 dx \\ &= \int_0^\epsilon \int_{S_r} (\frac{1}{2}R_{,ij}x^i x^j + O(r^3))u_\alpha^2 d\omega_r dr + O(\alpha^{n-2}) \\ &= \int_0^\epsilon (-Cr^2|W(P)|^2 + O(r^3))u_\alpha^2 r^{n-1} dr + O(\alpha^{n-2}), \end{aligned}$$

for some constant $C > 0$. A calculation of $\int_0^\epsilon r^k u_\alpha^2 r^{n-1} dr$ gives

$$A(\varphi) \leq \begin{cases} \lambda(S^n)||\varphi||_p^2 - C|W(P)|^2\alpha^4 + O(\alpha^5), & n > 6, \\ \lambda(S^n)||\varphi||_p^2 - C|W(P)|^2\alpha^4 \log(1/\alpha) + O(\alpha^4), & n = 6. \end{cases}$$

If M is not locally conformally flat, we can choose P such that $|W(P)|^2 > 0$, then $Y(\varphi^{p-2}g) < \lambda(S^n, \overline{g})$ for α sufficiently small and $n \geq 6$. Thus $\lambda(M, g) < \lambda(S^n, \overline{g})$.

2.1.6 Schoen's work and positive mass theorem

Aubin's result is limited to dimension ≥ 6 because these are the dimensions in which the local conformal geometry contains enough information to solve the problem. In the remaining cases the problem becomes a global one, which was solved by Schoen [Sch84] by using positive mass theorem (see [SY79a], [SY79b], [SY81a] and [SY81b]).

Definition 2.9 *Suppose (M, g) is a compact Riemannian manifold with $\lambda(M, g) > 0$. For $P \in M$ we define the metric $\hat{g} = G^{p-2}g$ on $\hat{M} = M - \{P\}$, where $G = (n-2)\omega a\Gamma_P$ and Γ_P is the Green function of $\Box$ at P on (M, g). The manifold $(\hat{M}, \hat{g})$ together with the natural map $\sigma : M - \{P\} \to \hat{M}$ is called the* **stereographic projection** *of M from P.*

Note that if $\lambda(M, g) > 0$, at each point $P \in M$ the Green function Γ_P for $\Box$ exists and is strictly positive.

Under the classic Stereographic projection, the Euclidean metric pulls back to a metric $\hat{g}$ on $S^n - \{P\}$, conformal to the standard metric $\overline{g}$, with zero scalar curvature. Therefore $\hat{g} = G^{p-2}\overline{g}$ where G is a multiple of the Green function for $\Box$ at P on S^n.

In general $(\hat{M}, \hat{g})$ is not flat but asymptotically flat . A Riemannian manifold (M, g) is called **asymptotically flat of order** $\tau > 0$ if there exists a decomposition $M = M_0 \cup M_\infty$ (with M_0 compact) and a diffeomorphism $M_\infty \leftrightarrow R^n - B_R$ for some $R > 0$, satisfying:

$$g_{ij} = \delta_{ij} + O(\rho^{-\tau}), \partial_k g_{ij} = O(\rho^{-\tau-1}), \partial_k\partial_l g_{ij} = O(\rho^{-\tau-2}),$$

as $\rho = |z| \to \infty$ in the coordinates z^i induced on M_∞. The coordinates z^i are called **asymptotic coordinates**.

The definition apparently depends on the choice of asymptotic coordinates. However, the asymptotically flat structure is determined by the metric alone.

The expansion of the Green function is important. By using it, R. Schoen proved the following. This completed the solution of the Yamabe problem.

Theorem 2.10 *([Sch84]) Let M^n be a compact Riemannian manifold with $n \geq 3$. If (M, g) is not conformally diffeomorphic to S^n, then the Sobolev quotient $\lambda(M, g)$ is strictly less then $\lambda(S^n, \overline{g})$.*

Now we go to the expansion of the Green function. For convenience we adopt the following notation.

NOTATION. We write $f = O'(r^k)$ to mean $f = O(r^k)$ and $\nabla f = O(r^{k-1})$. O'' is defined similarly.

In conformal normal coordinates $\{x^i\}$ at P, G has an asymptotic expansion

$$G(x) = r^{2-n}(1 + \sum_{k=4}^{n} \psi_k(x)) + c \log r + O''(1),$$

where $r = |x|$, ψ_k is in space of homogeneous polynomials of degree k, and the log term appears only if n is even. The leading terms are:

(a) if $n = 3, 4, 5$, or M is conformally flat in a neighborhood of P,

$$G = r^{2-n} + A + O''(r), A \textit{ is some constant};$$

(b) if $n = 6$,

$$G = r^{2-n} - \frac{1}{288a}|W(P)|^2 \log r + O''(1);$$

(c) if $n \geq 7$,

$$G = r^{2-n}[1 + \frac{1}{12a(n-4)}(\frac{r^4}{12(n-6)}|W(P)|^2 - R_{,ij}(P)x^i x^j r^2)] + O''(r^{7-n}).$$

The asymptotically flat structure of $\hat{g}$ can be derived immediately from the expression of the Green function.

Let $\{x^i\}$ be conformal normal coordinates on a neighborhood U of P and define **"inverted conformal coordinates"** $z^i = r^{-2}x^i$ on $U - \{P\}$. With $\rho = |z| = r^{-1}$ we have

$$\partial/\partial z^i = \rho^{-2}(\delta_{ij} - 2\rho^{-2}z^i z^j)\partial/\partial x^j.$$

If we write $\gamma = r^{n-2}G$, the components of $\hat{g}$ in z-coordinates are

$$\begin{aligned} \hat{g}_{ij}(z) &= \gamma^{p-2}\rho^4 g(\partial/\partial z^i, \partial/\partial z^j) \\ &= \gamma^{p-2}(\delta_{ik} - 2\rho^{-2}z^i z^k)(\delta_{jl} - 2\rho^{-2}z^j z^l)g_{kl}(\rho^{-2}z) \\ &= \gamma^{p-2}(\delta_{ij} + O''(\rho^{-2})). \end{aligned}$$

Therefore in inverted conformal normal coordinates it has the expansion

$$\hat{g}_{ij}(z) = \gamma^{p-2}(z)(\delta_{ij} + O''(\rho^{-2})),$$

where, in the three cases

(a) if $n = 3, 4, 5$, or M is conformally flat in a neighborhood of P,

$$\gamma(z) = 1 + A\rho^{2-n} + O''(\rho^{1-n}), A \textit{ is some constant};$$

(b) if $n = 6$,

$$\gamma(z) = 1 + \frac{1}{288a}|W(P)|^2\rho^{-4}\log\rho + O''(\rho^{-4});$$

(c) if $n \geq 7$,

$$\gamma(z) = 1 + \frac{1}{12a(n-4)}\rho^{-6}(\frac{\rho^2}{12(n-6)}|W(P)|^2 - R_{,ij}(P)z^iz^j) + O''(\rho^{-5}).$$

So $(\hat{M}, \hat{g})$ is an asymptotically flat manifold as a stereographic projection of $M - P$ to $\hat{M}$ with metric $\hat{g} = G^{p-2}g$. Now we can choose the inverted conformal normal coordinates as an asymptotically flat coordinates. To see what happens in this setting, we choose (R^n, ds^2) as a simple model.

For $\alpha > 0$ let $u_\alpha = (\frac{|x|^2+\alpha^2}{\alpha})^{(2-n)/2}$ as before and $(S^n, \overline{g})$ be the standard sphere. Since the inverted conformal normal coordinates $z^i = r^{-2}x^i$, by (2.6),

$$\begin{aligned}
\delta_\alpha^*\rho^*\overline{g} &= 4u_\alpha^{p-2}dx^2 \\
&= 4(\frac{\alpha^2 + |z|^{-2}}{\alpha})^{-2}|z|^{-4}dz^2 \\
&= 4(\frac{|z|^2 + \alpha^{-2}}{\alpha^{-1}})^{-2}dz^2 \\
&= 4u_{\alpha^{-1}}(z)^{p-2}dz^2.
\end{aligned}$$

Now it's natural to choose a test function φ on $(\hat{M}, \hat{g})$ in the following way. Fix a large radius $R > 0$, let $\rho(z) = |z|$ in inverted conformal normal coordinates, and let $\hat{M}_\infty = \{\rho > R\}$. Define φ on $\hat{M}$ by $\varphi(z) = u_\alpha(z)$, for $\rho(z) \geq R$ and $\varphi(z) = u_\alpha(R)$, for $\rho(z) \leq R$, with $\alpha >> R$ to be determined as we like.

Observe that, as $\alpha \to \infty$, $u_\alpha(z)$ becomes nearly constant for $|z| \leq R$ ($|x| \geq R^{-1}$), and so we can expect that the effect of replacing u_α as a constant inside radius R should become negligible. Moreover, the metric on $\hat{M}_\infty$ closely approximates the Euclidean metric, and so the Yamabe functional $Y(\varphi^{p-2}\hat{g})$ should become close to $\lambda(S^n, \overline{g})$.

Since φ is a function of the radial variable ρ alone, the behavior of $Y(\varphi^{p-2}\hat{g})$ as $\alpha \to \infty$ depends on the "average" behavior of the metric $\hat{g}$ over large spheres. It is useful to introduce a number, which we call the "distortion coefficient" of $\hat{g}$, that measures this average behavior.

It is well known that the scalar curvature measures the deviation of volumes from the Euclidean case. To see this, let $\widetilde{g}$ denote any metric on a manifold M, and let $r = |x|$ in normal coordinates around a point $P \in M$. The ratio of the $\widetilde{g}$-volume of the geodesic sphere S_r around P to its Euclidean volume is given by the sphere density function (see [LP87])

$$\begin{aligned} h(r) &= \omega_r^{-1} \int_{S_r} d\widetilde{\omega}_r \\ &= \omega_r^{-1} \int_{S_r} (\widetilde{g}^{rr} det\widetilde{g})^{\frac{1}{2}} d\omega_r \\ &= 1 - \frac{1}{6n} Rr^2 + O(r^3), \end{aligned}$$

where $d\widetilde{\omega}_r$ is the volume element induced on S_r by $\widetilde{g}$.

On an asymptotically flat manifold we consider the same function $h(\rho)$ for large value $\rho = |z|$. In particular, on the manifold $(\hat{M}, \hat{g})$ obtained by stereographic projection with the inverted conformal normal coordinates z^i, we have $\hat{g}^{\rho\rho} = \gamma^{2-p}$ and $\det\hat{g} = \gamma^{2p}$. Thus

$$h(\rho) = \omega_\rho^{-1} \int_{S_\rho} (\hat{g}^{\rho\rho} det\hat{g})^{\frac{1}{2}} d\omega_\rho = \omega_\rho^{-1} \int_{S_\rho} \gamma^{(p+2)/2} d\omega_\rho.$$

The expansion of γ then gives an asymptotic expansion of $h(\rho)$ as $\rho \to \infty$:

$$h(\rho) = \begin{cases} 1 + (\mu/k)\rho^{-k} + O''\rho^{-k-1}), & n \neq 6, \\ 1 + (\mu/4)\rho^{-4} \log\rho + O''(\rho^{-4}), & n = 6. \end{cases}$$

Since the $(n-1)$-form $d\omega_\rho/\omega_\rho$ is homogeneous of degree zero,

$$\frac{a}{2} \int_{S_\rho} \partial_\rho \gamma \frac{d\omega_\rho}{\omega_\rho} = h'(\rho) + O(\rho^{-2k-1}) = \begin{cases} -\mu\rho^{-k-1} + O(\rho^{-k-2}), & n \neq 6 \\ -\mu\rho^{-5} \log\rho + O(\rho^{-5}), & n = 6 \end{cases}$$

for some constant μ and $k \geq 1$. We call the constant μ, defined using inverted conformal normal coordinates, the **distortion coefficient** of $\hat{g}$. Its geometric meaning at infinity is analogous to that of the scalar curvature at a finite point. It is this constant that determines the values of $\lambda(\hat{M}, \hat{g})$ for large α.

Proposition 2.11 *(see [LP87]) Let φ be defined as above. There are positive constants C and k such that*

$$A(\varphi) \leq \lambda(S^n, \overline{g})||\varphi||_p^2 - C\mu\alpha^{-k} + O(\alpha^{-k-1}),$$

if $n \neq 6$ or M is conformally flat near P;

$$A(\varphi) \leq \lambda(S^n, \overline{g})||\varphi||_p^2 - C\mu\alpha^{-4} \log\alpha + O(\alpha^{-4}),$$

if $n = 6$ and M is not conformally flat near P. Thus if $\mu > 0$, φ can be chosen so that $Y(\varphi^{p-2}\hat{g}) < \lambda(S^n, \overline{g})$.

This proposition reduces the solution of the Yamabe problem in the case $\lambda(M, g) > 0$ to determining the sign of μ. Indeed,

$$\lambda(M, g) = \inf_{\psi \in C_0^\infty(\hat{M})} \frac{A(\psi)}{||\psi||_p^2},$$

and so approximating our test function φ by a function $\psi \in C_0^\infty(\hat{M})$, we find that $\lambda(M,g) < \lambda(S^n, \overline{g})$ if $\mu > 0$. It establishes:

Theorem 2.12 *If (M,g) is a compact Riemannian manifold of dimension $n \geq 3$ with $\lambda(M,g) > 0$, then $\lambda(M,g) < \lambda(S^n, \overline{g})$ if there is a generalized stereographic projection $\hat{M}$ of M with strictly positive distortion coefficient μ.*

The positivity of μ is proved by using positive mass theorem. We follow [LP87] to give a rough description of it.

Definition 2.13 *Given an asymptotically flat Riemannian manifold (M,g) with asymptotic coordinates z^i, define the* **mass** *as follows:*

$$m(g) = \lim_{R\to\infty} \omega_R^{-1} \int_{S_R} \nu \lrcorner dz,$$

if the limit exists, where ν is the mass-density vector defined on M_∞:

$$\nu = (\partial_i g_{ij} - \partial_j g_{ii})\partial_j.$$

The distortion coefficient μ is related to the mass.

Theorem 2.14 *Let $\hat{M}$ be the stereographic projection of M from $P \in M$, and μ the distortion coefficient computed with respect to inverted conformal normal coordinates. If $n < 6$ or M is conformally flat in a neighborhood of P, then $\mu = \frac{1}{2}m(\hat{g})$.*

The weighted C^k space $C^k_\beta(M)$ as the set of C^k functions u for which the norm

$$||u||_{C^k_\beta} = \sum_{i=0}^{k} \rho^{-\beta+i}|\nabla^i u|$$

is finite. The weighted *Hölder* space $C^{k,\alpha}_\beta(M)$ is defined for $0 < \alpha < 1$ as the set of $u \in C^k_\beta(M)$ for which the norm

$$||u||_{C^{k,\alpha}_\beta} = ||u||_{C^k_\beta} + \sup_{x,y}\{(\min\{\rho(x),\rho(y)\})^{-\beta+k+\alpha}\frac{|\nabla^k u(x) - \nabla^k u(y)|}{|x-y|^\alpha}\}$$

is finite.

For $\tau > (n-2)/2$ we define $\mathcal{M}_\tau$ to be the set of all C^∞ metrics on M such that (in some asymptotic coordinates)

$$g_{ij} - \delta_{ij} \in C^{1,\alpha}_{-\tau}(\mathcal{M}_\infty).$$

The positive mass theorem says the mass is positive therefore the distortion coefficient is positive.

Theorem 2.15 *(positive mass theorem) Let (M, g) be an asymptotically flat Riemannian manifold of dimension $n \geq 3$ with metric $g \in \mathcal{M}_\tau$, $\tau > (n-2)/2$, and nonnegative scalar curvature. Then its mass $m(g)$ is nonnegative, with $m(g) = 0$ if and only if (M, g) is isometric to $\mathbb{R}^n$ with its Euclidean metric.*

2.2 The Yamabe flow

The Riemannian Yamabe flow is defined by

$$\frac{\partial g}{\partial t} = (r_g - R_g)g, \tag{2.8}$$

where r_g denotes the average of scalar curvature, i.e. $r_g = \frac{\int_M R_g dv_g}{\int_M dv_g}$. Since the conformal class of g is preserved along the Yamabe flow, we may write $g = u^{\frac{4}{n-2}} g_0$, where g_0 is a fixed background metric on M and u is a positive function. The Yamabe flow can be rewritten as (up to a constant scale of time)

$$\frac{\partial u}{\partial t} = u^{-\frac{4}{n-2}} \triangle_{g_0} u - bR_{g_0} u^{1-\frac{4}{n-2}} + b r_g u,$$

with $b = a^{-1} = \frac{n-2}{4(n-1)}$.

The long-time existence was first obtained by R. Hamilton in an unpublished paper. In case the initial metric has negative scalar curvature, Hamilton proved the convergence of the flow. For the special case that the initial metric has positive Ricci curvature and is locally conformally flat, B. Chow obtained the convergence [Cho92].

In 1994, R. Ye [Ye94] using the Alexandrov reflection principle, obtained the convergence of flow (2.8) if M is locally conformally flat. In particular, a uniform a priori C^1-bounds for the solution of the Yamabe flow (2.8) is obtained on any locally conformally flat manifold. As $t \to \infty$ the associated metrics $g(t) = u(t)^{\frac{4}{n-2}} g_0$ then smoothly converge to a limit metric $g_\infty = u_\infty^{\frac{4}{n-2}} g_0$ of constant scalar curvature.

A recent work by H. Schwetlick and M. Struwe [SS03] proved that the Yamabe flow (2.8) converges in dimensions $3 \leq n \leq 5$ under the assumption that the initial metric satisfies $r_{g_0} \leq (\lambda(M, g)^{\frac{n}{2}} + \lambda(S^n, \overline{g})^{\frac{n}{2}})^{\frac{2}{n}}$. Under this assumption, it is shown that any possible singularity consists of a single bubble only, and the formation of singularity of this kind is ruled out using the positive mass theorem.

More recently, S. Brendle settled down almost all cases of the Yamabe flow ([Bre05]).

The Yamabe flow (2.8) on a surface was considered by Hamilton [Ham88]. In this special case, it's in fact the Ricci flow on a surface. In this case Hamilton developed a Harnack inequality and an entropy formula therefore a uniform bound of the scalar curvature is obtained. He proved:

Theorem 2.16 *([Ham88]) Let (M, g) be a compact oriented Riemannian surface.*

(i) If M is not diffeomorphic to the 2-sphere S^2, then any metric g converges to a constant curvature metric under the Yamabe flow (2.8).

(ii) If M is diffeomorphic to S^2, then any metric g with positive Gauss curvature on S^2 converges to a metric of constant curvature under the Yamabe flow.

In an extensive work by B. Chow, he proved:

Theorem 2.17 *([Cho91]) For any metric g on S^2, then under the Yamabe flow, the Gauss curvature becomes positive in finite time.*

Combine them it yields:

Corollary 2.18 *For any metric g on a Riemann surface, under the Yamabe flow (2.8) g converges to a metric of constant curvature.*

A generalization of the Yamabe flow on manifolds with boundary was considered by S. Brendle [Bre02].

We give a brief introduction to the works of Ye, Schwetlick, Struwe and Brendle.

2.2.1 Ye's approach by using the heat equation

R. Ye [Ye94] applied the flow method to deal with the Riemannian Yamabe problem. It is a negative gradient flow of the Yamabe functional defined on a conformal class. Let (M, g) be a compact manifold of dimension $n \geq 3$. The Yamabe functional Y on $[g_0]$ is defined by

$$Y(g) = \frac{\int_M R_g dv_g}{(\int_M dv_g)^{\frac{n-2}{n}}}, \quad g \in [g_0],$$

where dv_g is the volume form of g and R_g denotes the scalar curvature of g.

By the first variation formula (2.3)

$$DY_g(h) = \frac{1}{V^{\frac{n-2}{n}}} \int_M < -R_{ij} + \frac{1}{2} R_g g_{ij} - \frac{n-2}{2n} r_g g_{ij}, h_{ij} >_g dv_g,$$

we see the gradient of Y at g is given by $\frac{n-2}{2n} V(g)^{-\frac{2}{p}}(R_g - r_g)g$. The negative gradient flow of Y at g is hence given by

$$\frac{\partial g}{\partial t} = \frac{n-2}{2n} V(g)^{-\frac{2}{p}}(r_g - R_g)g.$$

We can see this flow preserves the volume. In fact,

$$\frac{dV}{dt} = \int_M \frac{d}{dt} dv_g = \frac{n-2}{4} \int_M V^{-\frac{2}{p}}(r_g - R_g) dv_g = 0.$$

If we change time by a constant scale, it becomes the Yamabe flow (2.8).

If the flow exists for all time and converges smoothly as $t \to \infty$, then the limit metric would have constant scalar curvature. Hence, Yamabe flow should be an effective tool to find metrics of constant scalar curvatures in a given conformal class. Indeed, it was originally conceived to attack the Yamabe problem. The significance of the Yamabe flow is that it is a natural geometric deformation leading to metric of constant scalar curvature.

We say that $[g_0]$ scalar positive, scalar negative, or scalar flat, if $[g_0]$ contains a metric of positive, negative, or identically zero scalar curvature respectively. It is well known that these three cases are mutually exclusive and exhaust all possibilities.

In fact, let u be the first eigenfunction of the conformal Laplacian $\Box = -a\triangle^0 + R_{g_0}$ on M with eigenvalue

$$\lambda_1 = \inf_{u>0} \frac{\int_M (a|\nabla u|_0^2 + R_0 u^2) dv_0}{\int_M u^2 dv_0}.$$

λ_1 has the same sign with $\lambda(M, g_0)$. Denote $g = u^{\frac{4}{n-2}} g_0$. Since $-a\triangle^0 u + R_0 u = \lambda_1 u = R_g u^{p-1}$, we have $R_g = \lambda_1 u^{2-p}$. Therefore scalar positive, scalar negative, and scalar flat if and only if $\lambda(M, g_0)$ is positive, negative, and zero.

Fix a background metric $g_0 \in [g_0]$ and write $g = u^{\frac{4}{n-2}} g_0$ with u denoting a positive function. Then (2.8) can be written in the equivalent form

$$\begin{aligned}
\frac{\partial u^{\frac{4}{n-2}}}{\partial t} &= (r_g - R_g) u^{\frac{4}{n-2}} \\
&= [r_g - (-a\triangle_{g_0} u + R_{g_0} u) u^{1-p})] u^{\frac{4}{n-2}} \\
&= a u^{-1} \triangle_{g_0} u - R_{g_0} + r_g u^{p-2},
\end{aligned}$$

where we have used the transformation law (2.1)

$$R_g = (-a\triangle_{g_0} u + R_{g_0} u)u^{1-p}.$$

Or, in consequence of changing time by a constant scale,

$$\frac{\partial u}{\partial t} = u^{-\frac{4}{n-2}}\triangle_{g_0} u - bR_{g_0} u^{1-\frac{4}{n-2}} + br_g u, \tag{2.9}$$

with $b = a^{-1} = \frac{n-2}{4(n-1)}$. Note that in most cases we make no difference between (2.8) and (2.9).

In the following we identify $R_0 = R_{g_0}, \quad R = R_g, \quad r = r_g$ and $\quad dv = dv_g$.

We can see flow (2.8) has the following properties:

(i)

$$\frac{dV}{dt} = \int_M \frac{\partial}{\partial t} dv = \frac{n}{2}\int_M (r-R)dv = 0, \tag{2.10}$$

(ii)

$$\frac{dY}{dt} = -\frac{n-2}{2} V^{-2/p} \int_M (r-R)^2 dv. \tag{2.11}$$

The Yamabe flow has short time existence. It follows from the linear theory and the implicit function theorem.

Proposition 2.19 *For each $\delta > 0$ and $\Lambda > 0$ there is some $T > 0$ depending on δ, Λ, n and the background metric metric g_0 with the following properties. If u_0 is a positive smooth function on M satisfying $u_0 \geq \delta$, $||u_0||_{C^3(M,g_0)} \leq \Lambda$(norm w.r.t. g_0), then the Yamabe flow with initial data u_0 has a unique positive smooth solution on the maximum time interval $[0,T)$.*

The main theorem in [Ye94] is the following.

Theorem 2.20 *([Ye94]) Assume that $[g_0]$ is scalar positive. Assume in addition that $(M,[g_0])$ is locally conformally flat. Then for any given initial metric in $[g_0]$, the flow has a unique smooth solution on the time interval $[0,\infty)$. Moreover, the solution metric g converges smoothly to a unique limit metric of constant scalar curvature as $t \to \infty$.*

This theorem finds a Yamabe solution on locally conformally flat manifold. It's interesting to note that the standard sphere S^n is included in Theorem 2.20. It has been a tradition to emphasize the difference between S^n and other manifolds in the context of the Yamabe problem, it might appear unexpected that the Yamabe flow on S^n always converges. It's proof depends on the following Harnack inequality for solution u.

Lemma 2.21 *([Ye94]) Assume that $(M,[g_0])$ is locally conformally flat and that $[g_0]$ is scalar positive. Choose a background metric $g_0 \in [g_0]$. If g is a solution of (2.8) with initial metric $g^0 \in [g_0]$ and u denotes the corresponding solution of (2.9), then*

$$\sup_t \frac{|\nabla_{g_0} u|}{u} \leq C, \tag{2.12}$$

where C is a positive constant depending only on g^0, g_0, and the conformal properties of $(M,[g_0])$. For each t, integrating (2.12) along a shortest geodesic between a maximum point and a minimum point of $u(\cdot,t)$ yields

$$\inf_t u \geq c \sup_t u \tag{2.13}$$

for some $c > 0$.

Since the Yamabe flow preserves the volume and

$$V(t) = \int_M u^p dv_{g_0},$$

the Harnack inequality (2.13) implies that u is uniformly bounded from above and away from zero. Denote T^* as the maximal time such that flow (2.9) has a smooth solution. Standard linear theory and bootstrapping then yield smooth estimates for u on $M \times [min(1,T^*/2),T^*)$. It follows that $T^* = \infty$, since otherwise we would be able to extend u beyond T^* by proposition 2.19. By L. Simon's [Sim83] general results, g converges smoothly to a unique limit g_∞ as $t \to \infty$. On the other hand, from (2.11)

$$\frac{dY}{dt} = -\frac{n-2}{2} V^{-2/p} \int_M (r-R)^2 dv,$$

we have

$$\int_0^\infty \int_M (R-r)^2 dv dt < \infty. \tag{2.14}$$

So the limit metric g_∞ has constant scalar curvature.

Respect to the scalar negative and flat cases, the following theorem has been proved in [Ye94].

Theorem 2.22 *([Ye94]) Assume that $[g_0]$ is either scalar negative or scalar flat. Then for any given initial metric in $[g_0]$, the solution of the Yamabe flow exists for all time and converges smoothly to a unique limit of constant scalar curvature as $t \to \infty$.*

First, we discuss the scalar negative case. Choose a background metric g_0 such that $R_{g_0} < 0$. Let $g^0 \in [g_0]$ be an initial metric and g the solution of (2.8) with $g(0) = g^0$ on a maximal time interval $[0, T^*)$. Apply the maximum principle to (2.9)

$$\frac{\partial u}{\partial t} = u^{-\frac{4}{n-2}} \triangle_{g_0} u - bR_{g_0} u^{1-\frac{4}{n-2}} + br_g u,$$

we get

$$\frac{d}{dt} u_{min}(t) \geq b \min |R_{g_0}| u_{min}^{1-\frac{4}{n-2}} + bru_{min}, \tag{2.15}$$

where $u_{min}(t) = \min_t u$. Since

$$r = YV^{-\frac{2}{n}}$$

and

$$Y(g) = \frac{\int_M R_g dv_g}{(\int_M dv_g)^{\frac{n-2}{n}}} = \frac{\int_M (a|\nabla u|^2_{g_0} + R_{g_0} u^2) dv_{g_0}}{(\int_M u^{\frac{2n}{n-2}} dv_{g_0})^{\frac{n-2}{n}}} \geq -\alpha$$

for some constant $\alpha > 0$. Then we have

$$u_{min}(t) \geq \min\{u_{min}(0), (\alpha^{-1} \min |R_{g_0}| V^{2/n})^{\frac{n-2}{4}}\}. \tag{2.16}$$

On the other hand, the maximum principle also implies

$$\frac{d}{dt} u_{max}(t) \leq -b(\min R_{g_0}) u_{max}^{1-\frac{4}{n-2}} + bru_{max}, \tag{2.17}$$

where $u_{max}(t) = \max_t u$. By (2.11), $r \leq r(g^0)$. Consequently, we can assume

$$\frac{d}{dt} u_{max}(t) \leq cu_{max}(t),$$

where c depends only on n, α, $\min R_{g_0}$ and $r(g^0)$. Therefore,

$$u_{max}(t) \leq u_{max}(0) e^{ct}. \tag{2.18}$$

The estimates (2.16) and (2.18) imply $T^* = \infty$. Indeed, if $T^* < \infty$, then by (2.16) and (2.18) u would be uniformly bounded from above and away from zero on $[0, T^*)$. Then the solution would extend beyond T^*.

We claim that r will eventually become negative, even if it may not be so at the start. In fact, if r remains nonnegative always, then (2.15) would imply

$$\frac{d}{dt} u_{min}(t) \geq b \min |R_{g_0}| u_{min},$$

whence $u_{min}(t)$ approaches infinity as $t \to \infty$. This contradicts the constancy of volume. Choosing a later time as the time origin, we may assume $r(0) < 0$. Then $r \leq r(0)$ by (2.11). Hence (2.17) yields

$$u_{max}(t) \leq max\{u_{max}(0), (|r(0)|^{-1} \max |R_{g_0}|)^{\frac{n-2}{4}}\},$$

which together with (2.16) implies that u is uniformly bounded from above and away from zero. Therefore we obtain the uniform smooth estimates for u. By (2.14) we can get a limit g_∞ of g such that g_∞ has constant negative scalar curvature.

Next we treat the scalar flat case. Choose the background metric g_0 such that $R_{g_0} \equiv 0$. Note that r can never be negative. This is because

$$r_g = \frac{\int_M (a|du|^2_{g_0} + R_{g_0} u^2) dv_{g_0}}{\int_M u^{\frac{2n}{n-2}} dv_{g_0}} = \frac{\int_M a|du|^2_{g_0} dv_{g_0}}{\int_M u^{\frac{2n}{n-2}} dv_{g_0}} \geq 0.$$

If r is zero at the start, it remains so. Formula

$$\frac{dr}{dt} = -\frac{n-2}{2V} \int_M (r - R)^2 dv$$

implies that R has to be identically zero for all time. Thus the solution of the Yamabe flow is constant in time.

Next we assume that $r(0) > 0$. (2.15) implies

$$\frac{d}{dt} u_{min}(t) \geq bru_{min}.$$

Thus,

$$\log \frac{u_{min}(t)}{u_{min}(0)} \geq b \int_0^t r d\tau.$$

In a similar way, (2.17) implies

$$\log \frac{u_{max}(t)}{u_{max}(0)} \leq b \int_0^t r d\tau.$$

Hence we obtain the Harnack inequality

$$u_{min}(t) \geq \frac{u_{min}(0)}{u_{max}(0)} u_{max}(t).$$

It follows that u exists for all time, and uniform smooth estimates of u hold. Another consequence is that $r \to 0$ as $t \to \infty$. By (2.14) we can get a limit g_∞ of g such that g_∞ has scalar curvature zero. ■

Ye [Ye94] also proved that the Yamabe flow converges smoothly to a unique limit of constant scalar curvature at exponential rate as $t \to \infty$ in the scalar negative and scalar flat case.

Ye [Ye94] gave a proof of the long-time existence of the Yamabe flow (2.8) using a comparison argument based on maximum principle.

Theorem 2.23 *([Ye94]) For any given initial metric, the flow (2.8) has a unique smooth solution on the time interval* $[0, \infty)$.

The proof relies on the following two lemmas.

Lemma 2.24 *([Sac83] and [DiB83]) Let $C > 0$ and $\epsilon > 0$ be given constants. Let u be a positive smooth solution of (2.9) on some interval [0,T] with $u \leq C$, $|r| \leq C$, and $T > \epsilon$. Then the modulus of continuity of u on $M \times [\epsilon, T]$ can be estimated in terms of C,ϵ, n, and the background metric g_0.*

Lemma 2.25 *([Ye94]) Let u be a positive solution of (2.9) on some interval $[0, T)$ with $T < \infty$. Then u extends continuously to T and the extension is positive everywhere.*

The continuous extension of u is guaranteed by lemma 2.24, inequality (2.18) and the estimate $\alpha V^{-2/n} \leq r \leq r(0)$. Since the volume remains constant and is nonzero, the extension of u at T cannot vanish identically. Assume that $u(\cdot, T)$ is positive in a uniform neighborhood of p_0, a comparison argument based on the maximum principle implies that u is positive everywhere at time T.

Let u be a positive smooth solution of (2.9) on a maximal time interval $[0, T^*)$. If $T^* < \infty$, then u is uniformly bounded from above and away from zero. Hence u extends smoothly beyond T^*, contradicting the maximality of T^*. ■

2.2.2 Some recent works

Schwetlick and Struwe proved in [SS03] the following.

Theorem 2.26 *([SS03]) Let (M, g_0) be a smooth compact n-manifold, $3 \leq n \leq 5$. Assume that $R_0 = R_{g_0} > 0$ and $0 \leq \lambda(M, g_0) \leq r_0 \leq (\lambda(M, g_0)^{\frac{n}{2}} + \lambda(S^n, \overline{g}))^{2/n}$. Then there is a unique, global, smooth solution $u > 0$ of the Yamabe flow with initial data $u(0) = 1$ with associated metric $g(t) = u^{\frac{4}{n-2}} g_0$ such that the following holds true. Letting*

$$r_\infty = \lim_{t \to \infty} r(g(t)),$$

for any $p < \infty$ as $t \to \infty$ we have

$$R(t) \to r_\infty \quad in \quad L^p(M, g),$$

and for a suitable sequence $t_k \to \infty$ as $k \to \infty$ there holds

$$u(t_k) \to u_\infty \quad in \quad W^{2,p}(M, g_0),$$

for $p < \infty$, where $u_\infty > 0$ is a smooth function inducing a metric $g_\infty = u_\infty^{\frac{4}{n-2}} g_0$ of constant scalar curvature $R_\infty = r_\infty$.

The proof based mainly on estimates for suitable curvature integrals and precise concentration-compactness results rather than the maximum principle.

For $p \geq 1$, let $F_p(g) = \int_M |R - r|^p dv$. Choose a constant scale of time as in [SS03], now we have

$$\frac{\partial g}{\partial t} = \frac{4}{n-2}(r - R)g.$$

Therefore

$$\frac{\partial u}{\partial t} = (r - R)u = (a\triangle_0 u - R_0 u)u^{2-p} + ru$$

and

$$\frac{\partial R}{\partial t} = a\triangle_g R + \frac{4}{n-2}R(R - r).$$

By the evolution inequality

$$\begin{aligned}\frac{d}{dt}F_p(g) \leq & - \frac{4a(p-1)}{p}\int_M |\nabla(R-r)^{p/2}|_g^2 dv \\ & + 2pF_2(g)F_{p-1}(g) + \frac{4p}{n-2}rF_p(g) + \frac{|4p-2n|}{n-2}F_{p+1}(g),\end{aligned}$$

where $p > 1$, it was shown that

Lemma 2.27 *([SS03]) For any $0 < p < \infty$ there holds $F_p(g(t)) \to 0$ as $t \to \infty$.*

Let $B_r(x)$ denote the ball of radius r with respect to metric g_0 and with center at x. The following theorem reduces the behavior of the Yamabe flow to two cases, either convergence or concentration happens.

Theorem 2.28 *([SS03]) Let $g_k = u_k^{\frac{4}{n-2}} g_0$, where $0 < u_k \in C^\infty(M, g_0)$, $k \in N$, be a sequence of conformal metrics with unit volume and satisfying*

$$r_k = r(g_k) \leq C_0, \;\; F_p(g_k) \leq C_0$$

for all k and some $p > \frac{n}{2}$.

Then, either

(i) the sequence (u_k) is uniformly bounded in $W^{2,p}(M, g_0) \hookrightarrow L^\infty(M, g_0)$, or

(ii) there exists a subsequence (u_k) (relabelled) and finitely many points $x_1, \cdots, x_L \in M$ such that for any $r > 0$ and any $l \in \{1, \cdots, L\}$ there holds

$$\liminf_{k\to\infty}\left(\int_{B_r(x_l)} |R_k|^{\frac{n}{2}} dv_k\right)^{\frac{2}{n}} \geq \lambda(S^n, \overline{g}), \tag{2.19}$$

where $R_k = R_{g_k}$ and $dv_k = dv_{g_k}$; moreover, the sequence (u_k) is bounded in $W^{2,p}$ on any compact subset of $(M \setminus \{x_1, \cdots, x_L\}, g_0)$.

Concentration in sense of (2.19) also requires concentration of volume since

$$\begin{aligned}\left(\int_{B_r(x_l)} |R_k|^{\frac{n}{2}} dv_k\right)^{\frac{2}{n}} &\leq r_k\left(\int_{B_r(x_l)} dv_k\right)^{\frac{2}{n}} + \left(\int_{B_r(x_l)} |R_k - r_k|^{\frac{n}{2}} dv_k\right)^{\frac{2}{n}} \\ &\leq r_k\left(\int_{B_r(x_l)} dv_k\right)^{\frac{2}{n}} \\ &\quad + \left(\int_{B_r(x_l)} |R_k - r_k|^p dv_k\right)^{\frac{1}{p}}\left(\int_{B_r(x_l)} dv_k\right)^{\frac{2}{n}-\frac{1}{p}}.\end{aligned}$$

From the general concentration-compactness result, it was shown any possible singularity consists of a single bubble if $r_0 \leq (\lambda(M, g_0)^{\frac{n}{2}} + \lambda(S^n, \overline{g}))^{2/n}$.

For any $r > 0$ any $x_0 \in M$ and any $t_0 \leq t_1$, by $\frac{\partial g}{\partial t} = \frac{4}{n-2}(r - R)g$,

$$|Vol(B_r(x_0), g(t_1)) - Vol(B_r(x_0), g(t_0))| \leq c \int_{t_0}^{t_1} \int_{B_r(x_o)} |r - R| dv dt. \quad (2.20)$$

A Kazdan-Warner type condition is used to rule out the volume concentration and proved by using positive mass theorem.

A recent work by S. Brendle [Bre05] shows

Theorem 2.29 *([Bre05]) Suppose that M satisfies one of the following conditions:*

(i) $3 \leq n \leq 5$,

(ii) M is locally conformally flat,

(iii) The Weyl conformal curvature tensor of M is nowhere equal to 0.

Then, for all initial data, the Yamabe flow exists for all time and converges to a metric with constant scalar curvature.

In fact the following proposition is proved which ruled out the concentration of volume since (2.20).

Proposition 2.30 *([Bre05]) Suppose the same condition as in Theorem 2.29 is satisfied, then along the Yamabe flow*

$$\int_0^{\infty} (\int_M |R - r|^2 dv)^{\frac{1}{2}} dt \leq C.$$

It's worth to note that since there are no concentration then by Theorem 2.28 the function u has uniform bound from above and away from zero.

Chapter 3

The geometry of contact manifolds

In the chapter we introduce the geometry of contact manifolds. In particular, we assign metrics to make them as contact metric manifolds. Tanaka connection and Webster scalar curvature are also introduced in this chapter. One main topic of our interests is their conformal aspect. All the materials in this chapter are known, for more details one can see [YK84], [Bla76], [Bla02] and [Tan89].

3.1 Contact manifolds

In this section we introduce some basic concepts related to contact manifolds. We also present some examples.

Definition 3.1 (contact manifolds, contact distribution, contact form).

A (2n+1)-dimensional manifold M is called a contact manifold if it admits a 1-form θ' such that $\theta' \wedge (d\theta')^n \neq 0$ everywhere on M. The distribution $G = ker\theta' \subset TM$ is called a contact distribution. The contact distribution is invariant under a conformal transformation, i.e. $ker\theta = ker\theta'$ for $\theta = f(x)\theta'$, $f > 0$.

We fix a 1-form θ among $\{f\theta' : f > 0\}$, which is called a contact form associated with the contact distribution $ker\theta'$.

Since $\theta \wedge (d\theta)^n \neq 0$ is a volume form on M, a contact manifold is orientable. From $\theta \wedge (d\theta)^n \neq 0$, one can see the maximum dimension of an integral submanifold of the contact distribution G is only n.

Proposition 3.2 *For any contact form θ, there exists a unique vector field ξ such that*

$$L_\xi \theta = 0 \quad and \quad \theta(\xi) = 1,$$

so

$$i_\xi d\theta = 0,$$

where L_ξ *denotes the Lie derivative by* ξ *and* i_ξ *denotes the interior product operator by* ξ. ξ *is called the Reeb vector field (or characteristic vector field) associated with* θ.

Proof. Since $\theta \wedge (d\theta)^n$ is non-degenerate, $d\theta$ is a non-degenerate 2-form on G. On the other hand, since M is odd dimensional, $d\theta$ must be degenerate on TM. Therefore one obtains a line bundle l over M via the definition

$$l_p := \{v \in T_pM | d\theta(v, w) = 0, \forall w \in G\}.$$

The Reeb vector field ξ is given naturally by the section in l satisfying $\theta(\xi) = 1$. $L_\xi = i_\xi d + di_\xi$ implies $L_\xi \theta = i_\xi d\theta$, because $\theta(\xi) = 1$. By the definition of l one sees that $i_\xi d\theta = 0$. ■

A one dimensional integral submanifold of G is called a Legendre curve. A diffeomorphism f of M is called a contact transformation if $f^*\theta = \tau\theta$ for some nonzero function τ. If $\tau = 1$, f is called a strict contact transformation.

The local aspect of contact manifolds is simple. The classical Darboux theorem is the following.

Theorem 3.3 *About each point of a contact manifold* (M^{2n+1}, θ) *there exists local coordinates* $(x^1, \cdots, x^n, y^1, \cdots, y^n, z)$ *with respect to which*

$$\theta = dz - \sum_{i=1}^{n} y^i dx^i.$$

The name contact seems to be due to S. Lie and is natural in view of the simple example of Huygens' principle.

Consider $\mathbb{R}^2$ with coordinates (x, y). The classical notion of a "line element" of $\mathbb{R}^2$ is a point together with a non-vertical line through the point. Thus a line element may be regarded as a point in $\mathbb{R}^3$ determined by the point and the slope p of the line. Given a smooth curve C in the plane without vertical tangents, say $y = f(x)$, its tangent lines determine a curve in $\mathbb{R}^3$ with coordinates (x, y, p) which is a Legendre curve of the contact form $\theta = dy - pdx$. If now C is a wave front, by Huygens' principle the new wave front C_t at time t is the envelope of the circular waves centered at all the points of C, say of radius t taking the velocity of propagation to be 1. Corresponding to a point (x, y) on C, the point $(\overline{x}, \overline{y})$ on C_t lies on both the normal line and the circle of radius t centered at (x, y), i.e.

$$\overline{y} - y = -\frac{1}{p}(\overline{x} - x) \quad \textit{and} \quad (\overline{x} - x)^2 + (\overline{y} - y)^2 = t^2.$$

Thus $(\overline{x}-x)^2 = \frac{p^2t^2}{p^2+1}$, so depending on the direction of propagation, e.g. choosing the negative root, the transformation of $\mathbb{R}^3$ mapping (x, y, p) to

$$\overline{x} = x - \frac{pt}{\sqrt{p^2+1}}, \overline{y} = y + \frac{t}{\sqrt{p^2+1}}, \overline{p} = p,$$

maps C to C_t. Since $d\overline{y} - \overline{p}d\overline{x} = dy - pdx$, the transformation is a contact transformation. Moreover tangent wave fronts (curves) are mapped to tangent wave fronts (curves) and hence the name "contact".

The following examples are well known.

Example 3.4 *(1)* $\mathbb{R}^{2n+1}(x^1, \cdots, x^n, y^1, \cdots, y^n, z)$ *with contact form* $\theta = dz - \sum_{i=1}^n y^i dx^i$.

(2) J.Marinet [Mar71] proved that every compact orientable 3-manifold carries a contact structure.

(3) ([Gra59]) Let $i : M^{2n+1} \to \mathbb{R}^{2n+2}$ *be a smooth hypersurface immersed in* $\mathbb{R}^{2n+2}$ *and suppose that no tangent plane of* M^{2n+1} *pass through the origin of* $\mathbb{R}^{2n+2}$*, then* M^{2n+1} *has a contact structure.*

(4) The cotangent sphere bundle T_1^*M *and the tangent sphere bundle* T_1M*.*

(5) $T^*M \times \mathbb{R}$, T^3, T^5.

(6) CR manifolds and K-contact manifolds. We will discuss these two classes in more details later.

3.2 Contact metric manifolds

In this section we assign any contact manifold with a natural metric structure. Conformal transformation of contact forms is studied in this section.

Let (M, θ) be a contact manifold and ξ is the Reeb vector field. Let $J : TM \to TM$ be a tensor field of type (1,1) satisfying $J^2X = -X + \theta(X)\xi$. We call J an almost complex structure. It can be viewed as an almost complex structure defined on the contact distribution $G = ker\theta$ and extended to TM by $J\xi = 0$

Definition 3.5 (almost contact manifolds, contact metric manifolds) *A Riemannian metric* g *is called a* **compatible Riemannian metric** *with* J *if*

$$g(JX, JY) = g(X, Y) - \theta(X)\theta(Y).$$

Setting $Y = \xi$*, we have immediately*

$$g(X, \xi) = \theta(X).$$

Such (M, θ, J, g) is called an almost contact metric manifold. Furthermore, if the compatible Riemannian metric is given by

$$g(X, Y) = d\theta(X, JY) + \theta(X)\theta(Y),$$

we say g is an associated metric (with θ and J) and (M, θ, J, g) is called a contact metric manifold.

To any contact form θ, we can assign a contact metric structure (M, θ, J, g).

Proposition 3.6 *Let M be a 2n+1-dimensional manifold with a contact form θ. Then there exists a skew-symmetric tensor field J of type (1,1) and a Riemannian metric g such that*

$$J^2X = -X + \theta(X)\xi$$

and

$$g(X, Y) = d\theta(X, JY) + \theta(X)\theta(Y) = -d\theta(JX, Y) + \theta(X)\theta(Y)$$

for any vector fields X and Y on M.

Proof. We construct J and metric g as follows. Since $d\theta$ is a symplectic form on the contact distribution $G = ker\theta$, there exists a metric g' and an endomorphism J' defined on the contact distribution such that

$$g'(X, Y) = d\theta(X, J'Y) = -d\theta(J'X, Y)$$

and

$$J'^2 = -I.$$

Now extend J' to J by setting $J\xi = 0$. We define a Riemannian metric on TM by

$$\begin{aligned} g(X, Y) &= g'(X - \theta(X)\xi, Y - \theta(Y)\xi) + \theta(X)\theta(Y) \\ &= d\theta(X, JY) + \theta(X)\theta(Y). \end{aligned}$$

We can see also

$$J^2(X) = -X + \theta(X)\xi.$$

■

Mostly, we restrict our interests in contact metric manifolds (M, θ, J, g).

Proposition 3.7 *Let (M, θ, J, g) be a contact metric manifold. Concerning the Levi-Civita connection ∇ with respect to g, the following hold:*

(i) $\xi^j = g^{ij}\theta_i$;

(ii) $\nabla_\xi \xi = 0, \quad \nabla_\xi \theta = 0, \quad \xi^j \nabla_i \theta_j = 0$;

(iii) $div(\xi) = \nabla_r \xi^r = 0$.

Proof. (i)
$$g(\xi, \frac{\partial}{\partial x^i}) = \theta(\frac{\partial}{\partial x^i}) = \theta_i$$
implies
$$\xi^j = g^{ij}\theta_i.$$
(ii) Since for any vector field X,
$$0 = (L_\xi\theta)(X) = \xi g(X,\xi) - g(\nabla_\xi X - \nabla_X \xi, \xi) = g(X, \nabla_\xi \xi),$$
we have
$$\nabla_\xi \xi = 0.$$
Thus by (i) $\nabla_\xi \theta = 0$. It implies $\xi^j \nabla_i \theta_j = 0$.
(iii) Since $\theta \wedge (d\theta)^n$ is a volume form,
$$0 = L_\xi(\theta \wedge (d\theta)^n) = div(\xi)\theta \wedge (d\theta)^n.$$
Therefore
$$div(\xi) = \nabla_r \xi^r = 0.$$
■

Since $\nabla_\xi \xi = 0$, any integral curve of the Reeb vector field ξ is a geodesic.

Let (M, θ, J, g) be a contact metric manifold and let σ be a positive function on M. We consider a new contact form $\widetilde{\theta} = \sigma\theta$ and deduce the new contact metric structure $(M, \widetilde{\xi}, \widetilde{J}, \widetilde{g})$ corresponding to $\widetilde{\theta}$. It's natural to assume $\widetilde{J}$ is fixed in the following sense:

For each point x of M, the action of J and $\widetilde{J}$ are identical on $ker\theta_x = ker\widetilde{\theta}_x$, and $\widetilde{J}\widetilde{\xi} = 0$.

In the following, we will derive the explicit relations between (J, ξ, g) and $(\widetilde{J}, \widetilde{\xi}, \widetilde{g})$. $\widetilde{\xi}$ is defined by
$$(i) \quad \widetilde{\theta}(\widetilde{\xi}) = 1 \quad and \quad (ii) \quad i_{\widetilde{\xi}} d\widetilde{\theta} = 0.$$
By (i), there exist functions a^j such that
$$\widetilde{\xi} = \frac{1}{\sigma}\xi + a^j J(\frac{\partial}{\partial x^j}).$$
Denote $\sigma_i = \nabla_i \sigma$. Then (ii) induces
$$d(\sigma\theta)(\frac{1}{\sigma}\xi + a^j J(\frac{\partial}{\partial x^j}), \frac{\partial}{\partial x^i}) = 0.$$

Therefore

$$\frac{1}{\sigma}\xi(\sigma)\theta_i + a^j J(\frac{\partial}{\partial x^j})(\sigma)\theta_i - \frac{1}{\sigma}\sigma_i - \sigma a^j(g_{ij} - \theta_i\theta_j) = 0.$$

This can be rewritten as

$$\sigma a^j = \frac{1}{\sigma}\xi(\sigma)\xi^j + \xi^j a^k J(\frac{\partial}{\partial x^k})(\sigma) - \frac{1}{\sigma}\sigma^j + \sigma a^k\theta_k\xi^j.$$

Since $\xi^j J(\frac{\partial}{\partial x^j}) = 0$, multiplying the last identity by $J(\frac{\partial}{\partial x^j})$, we have

$$a^j J(\frac{\partial}{\partial x^j}) = -\frac{\sigma^j}{\sigma^2} J(\frac{\partial}{\partial x^j}).$$

Therefore,

$$\widetilde{\xi} = \frac{1}{\sigma}\xi - \frac{\sigma^j}{\sigma^2}J(\frac{\partial}{\partial x^j}) = \frac{1}{\sigma}\xi - \frac{1}{\sigma^2}J(\nabla\sigma).$$

Since

$$\widetilde{J} = J + \widetilde{J}(\xi)\theta,$$

one can assume

$$\widetilde{J}_i^j = J_i^j + v^j\theta_i.$$

Since

$$\widetilde{\xi}^i\widetilde{J}_i^j = 0 \quad \textit{and} \quad J_i^j J_k^i = -\delta_k^j + \theta_k\xi^j,$$

we have

$$\begin{aligned} 0 &= (\frac{1}{\sigma}\xi^i - \frac{\sigma^k}{\sigma^2}J_k^i)(J_i^j + v^j\theta_i) \\ &= \frac{1}{\sigma}v^j - \frac{\sigma^k}{\sigma^2}J_k^i J_i^j \\ &= \frac{1}{\sigma}v^j - \frac{\sigma^k}{\sigma^2}(-\delta_k^j + \theta_k\xi^j). \end{aligned}$$

Therefore

$$v^j = -\frac{1}{\sigma}(\sigma^j - \sigma^k\theta_k\xi^j)$$

and

$$\widetilde{J}_i^j = J_i^j - \frac{1}{\sigma}(\sigma^j - \sigma^k\theta_k\xi^j)\theta_i.$$

Denote

$$\widetilde{\xi} = \frac{1}{\sigma}(\xi + \zeta),$$

where

$$\zeta = -\frac{\sigma^j}{\sigma}J_j^k\frac{\partial}{\partial x^k} = -\frac{1}{\sigma}J(\nabla\sigma).$$

$$
\begin{aligned}
\widetilde{g}_{ij} &= d(\sigma\theta)(\frac{\partial}{\partial x^i}, \widetilde{J}\frac{\partial}{\partial x^j}) + \sigma^2\theta_i\theta_j \\
&= d\sigma \wedge \theta(\frac{\partial}{\partial x^i}, \widetilde{J}\frac{\partial}{\partial x^j}) + \sigma d\theta(\frac{\partial}{\partial x^i}, \widetilde{J}\frac{\partial}{\partial x^j}) + \sigma^2\theta_i\theta_j \\
&= \sigma_i\theta(\widetilde{J}\frac{\partial}{\partial x^j}) - \theta_i(\widetilde{J}\frac{\partial}{\partial x^j})(\sigma) \\
&\quad +\sigma d\theta(\frac{\partial}{\partial x^i}, J\frac{\partial}{\partial x^j} - \frac{1}{\sigma}(\sigma^l - \xi(\sigma)\xi^l)\theta_j\frac{\partial}{\partial x^l}) + \sigma^2\theta_i\theta_j,
\end{aligned}
$$

then we compute

$$\sigma_i\theta(\widetilde{J}\frac{\partial}{\partial x^j}) = 0,$$

$$
\begin{aligned}
-\theta_i(\widetilde{J}\frac{\partial}{\partial x^j})(\sigma) &= -\theta_i[J_j^l - \frac{1}{\sigma}(\sigma^l - \xi(\sigma)\xi^l)\theta_j]\sigma_l \\
&= -\theta_i J_j^l\sigma_l + \frac{1}{\sigma}(|\nabla\sigma|^2 - |\xi(\sigma)|^2)\theta_i\theta_j,
\end{aligned}
$$

$$\sigma d\theta(\frac{\partial}{\partial x^i}, J\frac{\partial}{\partial x^j}) = \sigma(g_{ij} - \theta_i\theta_j),$$

$$-\theta_j d\theta(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^l})(\sigma^l - \xi(\sigma)\xi^l) = \theta_j\sigma^l g_{ik}J_l^k,$$

$$||\zeta||^2 = \frac{1}{\sigma^2}(|\nabla\sigma|^2 - |\xi(\sigma)|^2).$$

Finally, we have

$$\widetilde{g}_{ij} = \sigma(g_{ij} - \theta_i\zeta_j - \theta_j\zeta_i) + \sigma(\sigma - 1 + ||\zeta||^2)\theta_i\theta_j.$$

One can observe the following relation holds,

$$\widetilde{g}^{ij} - \widetilde{\xi}^i\widetilde{\xi}^j = \sigma^{-1}(g^{ij} - \xi^i\xi^j).$$

This is because

$$g^{ij} - \xi^i\xi^j = < dx^i - < dx^i, \theta > \theta, dx^j - < dx^j, \theta > \theta >_g$$

and

$$d(\sigma\theta) = d\sigma \wedge \theta + \sigma d\theta.$$

Summarizing the above we obtain the following(see [Tan89]).

Proposition 3.8 *A conformal transformation $\theta \to \widetilde{\theta} = \sigma\theta$ of a contact form θ induces:*

(1) $\widetilde{\xi} = \frac{1}{\sigma}\xi - \frac{1}{\sigma^2}J(\nabla\sigma)$;

(2) $\widetilde{J}_i^j = J_i^j - \frac{1}{\sigma}(\sigma^j - \xi(\sigma)\xi^j)\theta_i$;

(3) $\widetilde{g}_{ij} = \sigma(g_{ij} - \theta_i\zeta_j - \theta_j\zeta_i) + \sigma(\sigma - 1 + ||\zeta||^2)\theta_i\theta_j$;

(4) $\widetilde{g}^{ij} - \widetilde{\xi}^i\widetilde{\xi}^j = \sigma^{-1}(g^{ij} - \xi^i\xi^j)$.

Now we define an operator $\triangle_P$, called sublaplacian, which is given by using the Laplacian $\triangle$ and ξ:

$$\triangle_P f = \triangle f - \xi\xi f = (g^{ij} - \xi^i\xi^j)\nabla_i\nabla_j f. \tag{3.1}$$

Since $\nabla_\xi \xi = 0$,

$$\nabla^2 f(\xi,\xi) = \nabla_\xi(\nabla f)(\xi) = \nabla_\xi\nabla_\xi f - (\nabla_\xi \xi) f = \xi\xi f.$$

Therefore, the sublaplacian operator is well defined.

Also define the sub-inner product for 1-forms

$$< \omega, \upsilon >_P = < \omega - \omega(\xi)\theta, \upsilon - \upsilon(\xi)\theta >_g . \tag{3.2}$$

Therefore

$$\begin{aligned} < df, df' >_P &= < df - \xi(f)\theta, df' - \xi(f')\theta >_g \\ &= (g^{ij} - \xi^i\xi^j)\nabla_i f \nabla_j f. \end{aligned}$$

Proposition 3.9 *The following identity holds:*

$$\int_M f'\xi\xi f dv = -\int_M \xi f \xi f' dv = \int_M f\xi\xi f' dv,$$

where dv denote the natural volume form $\theta \wedge d\theta^n$. Therefore,

$$\int_M f'\triangle_P f = \int_M f\triangle_P f' = -\int_M < df, df' >_P . \tag{3.3}$$

Proof.

$$\begin{aligned} 0 &= \int_M div(f'\xi f \xi) dv \\ &= \int_M f'\xi f div\xi dv + \int_M \xi f' \xi f dv + \int_M f'\xi\xi f dv, \end{aligned}$$

and

$$div\xi = 0$$

imply

$$\int_M f'\xi\xi f dv = -\int_M \xi f \xi f' dv.$$

∎

3.3 CR manifolds

CR manifolds are natural generalization of embedded hypersurfaces in complex manifolds. The geometry of CR manifolds is of great interest.

Let M be an orientable, real, $2n+1$-dimensional manifold. $\mathbb{C}TM$ is the complexification of tangent bundle TM. A **CR distribution** on M is given by a complex n-dimensional subbundle $T_{1,0}$ of $\mathbb{C}TM$, satisfying $T_{1,0} \cap T_{0,1} = \{0\}$, where $T_{0,1} = \overline{T}_{1,0}$. A general **CR manifold** is defined as a manifold together with a CR distribution $T_{1,0}$.

There is a unique subbundle G of TM such that its complexification $\mathbb{C}G$ can be written as $\mathbb{C}G = T_{1,0} + T_{0,1}$. G carries a natural complex structure map $J : G \to G$ given by $J(X + \overline{X}) = i(X - \overline{X})$ for $X \in T_{1,0}$. (G, J) is called the real expression of $T_{1,0}$. Conversely, $T_{1,0}$ can be expressed as $T_{1,0} = \{V - iJV; V \in G\}$. $T_{1,0}$ is also called the **holomorphic tangent bundle**.

Let $E \subset T^*M$ denote the real line bundle $G^{\perp}$. Because we assume M is oriented by its complex structure, E has a global nonvanishing section. Associated with each such section θ we define L_θ on G as

$$L_\theta(V, W) = \frac{1}{2} < d\theta, V \wedge JW > \quad for \quad V, W \in G.$$

Assume $T_{1,0}$ satisfy the partial integrability condition

$$[\Gamma(T_{1,0}), \Gamma(T_{1,0})] \subset \Gamma(\mathbb{C}G),$$

where $\Gamma(T_{1,0})$ denotes the space of all sections of $T_{1,0}$. The partial integrability condition is equivalent to that L_θ is symmetric.

In fact,

$$\begin{aligned}
&[X - iJX, Y - iJY] = [X, Y] - [JX, JY] - i([X, JY] + [JX, Y]) \in \Gamma(\mathbb{C}G) \\
&\Leftrightarrow \theta([X, JY]) = -\theta([JX, Y]) \quad and \quad \theta([X, Y]) = \theta([JX, JY]) \\
&\Leftrightarrow \theta([X, JY]) = -\theta([JX, Y]) \\
&\Leftrightarrow d\theta(X, JY) = d\theta(Y, JX),
\end{aligned}$$

for $X, Y \in G$.

The real symmetric bilinear form L_θ is called the Levi form of θ. We always assume M is **strictly pseudoconvex**, i.e. L_θ is positive definite for some suitable choice of θ. In this case θ defines a contact form on M. A **pseudo-hermitian structure** on M is a CR distribution together with a given contact form θ.

Since the partial integrability condition, the Levi form defined by

$$L_\theta(V, W) = \frac{1}{2} < d\theta, V \wedge JW >= d\theta(V, JW)$$

is compatible with J. Therefore $g = L + \theta \otimes \theta$ is an associated metric and (M, θ, J, g) is a contact metric manifold.

L_θ extends by complex linearity to $\mathbb{C}G$, and induces a hermitian form on $T_{1,0}$, which we write

$$L_\theta(V,\overline{W}) = -\frac{1}{2} < id\theta, V\wedge \overline{W} > \quad for \quad V,W\in T_{1,0}.$$

The inner product L_θ determines an isomorphism $G\cong G^*$, which in turn determines a dual form L^*_θ on G^*, which extends naturally to T^*M. It induces a norm $|\omega|_\theta$ on the space of real 1-forms ω,

$$|\omega|^2_\theta = L^*_\theta(\omega,\omega) = 2\sum_{j=1}^{n}|\omega(Z_j)|^2, \tag{3.4}$$

where $\{Z_1,\cdots,Z_n\}$ forms an orthonormal basis of $T_{1,0}$ with respect to the Levi form. The sublaplacian operator $\triangle_b$ is defined on real functions $u\in C^\infty(M)$ by

$$\int_M(-\triangle_b u)v\theta\wedge d\theta^n = \int_M L^*_\theta(du,dv)\theta\wedge d\theta^n, \quad for \quad all \quad v\in C^\infty_0(M).$$

Note that this normalization in (3.4) is different, i.e. $|\omega|^2_\theta = 2|\omega|^2_P$, where $<\cdot,\cdot>_P$ is the sub-inner product w.r.t. the contact metric manifold (M,θ,J,g) defined in (3.2). We also denote $<\cdot,\cdot>_\theta$ as $<\cdot,\cdot>_b$. Similarly, $\triangle_b = 2\triangle_P$.

The integrability condition of $T_{1,0}$ is defined as

$$[\Gamma(T_{1,0}),\Gamma(T_{1,0})]\subset\Gamma(T_{1,0}).$$

This condition is equivalent to $N_J + d\theta\otimes\xi = 0$ on G, where $N_J(X,Y) = J^2[X,Y]+[JX,JY]-J[JX,Y]-J[X,JY]$ is the Nijenhuis tensor. Note that a 3-dimension contact manifold is always a CR manifold since the complex dimension of $T_{1,0}$ is 1 therefore integrable.

We always assume the CR manifold M is integrable and strongly pseudoconvex.

A contact metric manifold (M,θ,J,g) is strongly pseudoconvex and partial integrable but not necessarily integrable. G is defined by $\theta=0$.

The most important example of integrable CR manifold is of course that induced by an embedding of M^{2n+1} in a complex manifold Ω of complex dimension $n+1$, in which case $T_{1,0} = T_{1,0}\Omega\cap\mathbb{C}TM$. If f is a defining function for M, then one choice of the contact form is $\theta = i(\overline{\partial}-\partial)f$.

3.4 The Webster scalar curvature

In this section we introduce the definition of the Webster scalar curvature defined on CR manifolds.

Let (M, θ) be a CR manifold. The contact form θ determines the Reeb vector field ξ as on contact metric manifolds. Choose $\{Z_\alpha\}_{\alpha=1}^n$ as any local frame of the holomorphic tangent bundle $T_{1,0}$. The **admissible coframe** dual to $\{Z_\alpha\}$ is the collection of $(1,0)$-forms $\{\theta^\beta\}$ defined by

$$\theta^\beta(Z_\alpha) = \delta_\alpha^\beta \quad \textit{and} \quad \theta^\beta(Z_{\overline{\alpha}}) = \theta^\beta(\xi) = 0.$$

Thus, $\{\xi, Z_\alpha, Z_{\overline{\beta}}\}$ forms a frame for $\mathbb{C}TM$, with dual coframe $\{\theta, \theta^\alpha, \overline{\theta}^\beta\}$.

Tanaka [Tan76] defined a natural linear connection on M according to the pseudohermitian structure on M. In literature, this connection is called Tanaka connection, Webster-Stanton connection or pseudohermitian connection.

With respect to an admissible coframe, the integrability condition $[T_{1,0}, T_{1,0}] \subset T_{1,0}$ and $\xi \lrcorner d\theta = 0$ imply that

$$d\theta = i h_{\alpha\overline{\beta}} \theta^\alpha \wedge \overline{\theta}^\beta, \tag{3.5}$$

with $h_{\alpha\overline{\beta}} = \overline{h}_{\beta\overline{\alpha}}$.

Webster [Web78] showed that there are uniquely determined 1-forms ω_α^β, τ^β on M satisfying

$$d\theta^\beta = \theta^\alpha \wedge \omega_\alpha^\beta + \theta \wedge \tau^\beta, \quad \omega_{\alpha\overline{\beta}} + \omega_{\overline{\beta}\alpha} = dh_{\alpha\overline{\beta}}, \quad \tau_\alpha \wedge \theta^\alpha = 0. \tag{3.6}$$

in which we have used the matrix $h_{\alpha\overline{\beta}}$ to raise and lower indices, e.g. $\omega_{\alpha\overline{\beta}} = \omega_\alpha^\gamma h_{\gamma\overline{\beta}}$. By (3.6), we can write

$$\tau_\alpha = A_{\alpha\gamma}\theta^\gamma, \tag{3.7}$$

with $A_{\alpha\gamma} = A_{\gamma\alpha}$.

The Tanaka connection denoted as ${}^*\nabla$ is defined in terms of holomorphic frame by

$${}^*\nabla Z_\alpha = \omega_\alpha^\beta \otimes Z_\beta, \quad {}^*\nabla\xi = 0. \tag{3.8}$$

Tanaka connection is not torsion free.

For a function f on M, we use notation

$$f_\alpha = Z_\alpha f, \quad f_{\overline{\alpha}} = Z_{\overline{\alpha}} f, \quad f_0 = \xi f,$$

thus

$$df = f_\alpha \theta^\alpha + f_{\overline{\alpha}} \overline{\theta}^\alpha + f_0 \theta.$$

The second covariant differential of f w.r.t. Tanaka connection is the 2-tensor with components

$$f_{\alpha\beta} = \overline{\overline{f}_{\overline{\alpha}\overline{\beta}}} = Z_\beta Z_\alpha f - \omega_\alpha^\gamma(Z_\beta) Z_\gamma f, \quad f_{\alpha\overline{\beta}} = \overline{\overline{f}_{\overline{\alpha}\beta}} = Z_{\overline{\beta}} Z_\alpha f - \omega_\alpha^\gamma(Z_{\overline{\beta}}) Z_\gamma f,$$
$$f_{0\alpha} = \overline{\overline{f}_{0\overline{\alpha}}} = Z_\alpha \xi f, \quad f_{\alpha 0} = \overline{\overline{f}_{\overline{\alpha}0}} = \xi Z_\alpha f - \omega_\alpha^\gamma(\xi) Z_\gamma f, \quad f_{00} = \xi^2 f.$$

(3.5) and (3.6) imply that

$$[Z_{\overline{\beta}}, Z_\alpha] = ih_{\alpha\overline{\beta}}\xi + \omega_\alpha^\gamma(Z_{\overline{\beta}}) Z_\gamma - \omega_{\overline{\beta}}^{\overline{\gamma}}(Z_\alpha) Z_{\overline{\gamma}},$$
$$[Z_\beta, Z_\alpha] = \omega_\alpha^\gamma(Z_\beta) Z_\gamma - \omega_\beta^\gamma(Z_\alpha) Z_\gamma,$$
$$[Z_\alpha, \xi] = A_\alpha^{\overline{\gamma}} Z_{\overline{\gamma}} - \omega_\alpha^\gamma(\xi) Z_\gamma,$$

therefore

$$f_{\alpha\overline{\beta}} - f_{\overline{\beta}\alpha} = ih_{\alpha\overline{\beta}} f_0, \quad f_{\alpha\beta} - f_{\beta\alpha} = 0, \quad f_{0\alpha} - f_{\alpha 0} = A_\alpha^{\overline{\gamma}} f_{\overline{\gamma}}.$$

The curvature of the Tanaka connection, expressed in terms of the coframe $\{\theta = \theta^0, \theta^\alpha, \overline{\theta}^\alpha\}$ is

$$\pi_\beta^\alpha = \overline{\pi_{\bar{\beta}}^{\bar{\alpha}}} = dw_\beta^\alpha - w_\beta^\gamma \wedge w_\gamma^\alpha,$$
$$\pi_0^\alpha = \pi_\alpha^0 = \pi_0^{\bar{\beta}} = \pi_{\bar{\beta}}^0 = \pi_0^0 = 0.$$

Webster showed that π_β^α can be written as

$$\pi_\beta^\alpha = R_{\beta\rho\overline{\sigma}}^\alpha \theta^\rho \wedge \bar{\theta}^\sigma + A_{\beta\;,\rho}^\alpha \theta^\rho \wedge \theta - A_{\beta\;,\bar{\rho}}^\alpha \bar{\theta}^\rho \wedge \theta + i\theta_\beta \wedge \tau^\alpha - i\tau_\beta \wedge \theta^\alpha,$$

where the coefficients satisfy

$$R_{\beta\bar{\alpha}\rho\bar{\sigma}} = \overline{R}_{\alpha\bar{\beta}\sigma\bar{\rho}} = R_{\bar{\alpha}\beta\bar{\sigma}\rho} = R_{\rho\bar{\alpha}\beta\bar{\sigma}}.$$

The Webster-Ricci tensor of (M, θ) is the hermitian form ρ on $T_{1,0}$ defined by

$$\rho(X, \overline{Y}) = R_{\alpha\bar{\beta}} X^\alpha Y^{\bar{\beta}},$$

where $X = X^\alpha Z_\alpha$, $Y = Y^\beta Z_\beta$, $R_{\alpha\bar{\beta}} = R_{\gamma\alpha\bar{\beta}}^\gamma$.

The Webster scalar curvature is

$$W = R_\alpha^\alpha = h^{\alpha\bar{\beta}} R_{\alpha\bar{\beta}}.$$

3.5 The generalized Webster scalar curvature

The Tanaka connection has been generalized by Tanno [Tan89] onto contact metric manifolds which makes it possible to consider the contact Yamabe problem.

Tanno [Tan89] generalized the Tanaka connection ${}^*\nabla$ defined on CR manifolds to any contact metric manifold (M, θ, J, g) given by

$$ {}^*\Gamma^i_{jk} = \Gamma^i_{jk} - \theta_j J^i_k - \nabla_j \xi^i \theta_k + \xi^i \nabla_j \theta_k := \Gamma^i_{jk} + W^i_{jk}, $$

where Γ^i_{jk} denotes the coefficients of the Levi-Civita connection ∇.

The torsion tensor ${}^*T$ and curvatures of ${}^*\nabla$ are given by

$$ {}^*T^i_{jk} = -\theta_j J^i_k + J^i_j \theta_k - \nabla_j \xi^i \theta_k + \nabla_k \xi^i \theta_j + \xi^i J^l_j g_{kl}, $$

$$ {}^*R^i_{jkl} = R^i_{jkl} + \nabla_k W^i_{lj} - \nabla_l W^i_{kj} + W^s_{lj} W^i_{ks} - W^s_{kj} W^i_{ls}, $$

$$ {}^*R_{jl} = R_{jl} + 2g_{jl} - 2\theta_j \theta_l - \theta_j R_{sl} \xi^s - R_{rjsl} \xi^r \xi^s - \nabla_r \theta_j \nabla_l \xi^r. $$

Let W also denote the generalized Webster scalar curvature on a contact metric manifold (M, θ, J, g) which is given by

$$ W = g^{ij}\ {}^*R_{ij}. $$

Therefore

$$ W = R - 2Ric(\xi, \xi) - \nabla_r \theta_j \nabla^j \xi^r + 4n. $$

Since

$$ \begin{aligned} \nabla_r \theta_j \nabla^j \xi^r &= \nabla_r (\theta_j \nabla^j \xi^r) - \xi^j \nabla_r \nabla_j \xi^r \\ &= -\xi^j \nabla_r \nabla_j \xi^r = -\xi^j \xi^k R^r_{rjk} \\ &= -Ric(\xi, \xi), \end{aligned} $$

it follows that

$$ W = R - Ric(\xi, \xi) + 4n. \tag{3.9} $$

Under the gauge transformation $\theta \to \widetilde{\theta} = \sigma\theta$, the transformation of the generalized Webster scalar curvature W is the following.

Proposition 3.10 *([Tan89]) Let $(\theta, g) \to (\widetilde{\theta} = u^{\frac{2}{n}}\theta, \widetilde{g})$ be a gauge transformation of contact metric structure. Then the transformation of the generalized Webster scalar curvature W is given by*

$$ \widetilde{W} = (-\mu \triangle_P u + W u) u^{-1-\frac{2}{n}}, \quad \mu = 2(2 + \frac{2}{n}). \tag{3.10} $$

Define

$$ \widetilde{A}^i_{jk} = \widetilde{\Gamma}^i_{jk} - \Gamma^i_{jk}, $$

where $\widetilde{\Gamma}^i_{jk}$ is the connection with respect to $\widetilde{\theta} = \sigma\theta$. We have the following formula.

Lemma 3.11 *([Tan89])*

$$ \widetilde{A}^i_{jl}(g^{jl} - \xi^j \xi^l) = \frac{n}{\sigma} \xi(\sigma) \xi^i - \frac{n}{\sigma} \sigma^i. $$

Proof. By the definition of $\widetilde{A}^i_{jk}$, we have

$$\widetilde{A}^i_{jl} = \frac{1}{2}\widetilde{g}^{is}(\nabla_j\widetilde{g}_{ls} + \nabla_l\widetilde{g}_{js} - \nabla_s\widetilde{g}_{jl});$$

$$\begin{aligned}
\widetilde{A}^i_{jl}(g^{jl} - \xi^j\xi^l) &= \frac{1}{2}\widetilde{g}^{is}(\nabla_j\widetilde{g}_{ls} + \nabla_l\widetilde{g}_{js} - \nabla_s\widetilde{g}_{jl})\sigma(\widetilde{g}^{jl} - \widetilde{\xi}^j\widetilde{\xi}^l) \\
&= \sigma\widetilde{g}^{is}\nabla_l\widetilde{g}_{js}(\widetilde{g}^{jl} - \widetilde{\xi}^j\widetilde{\xi}^l) - \frac{1}{2}\sigma\widetilde{g}^{is}\nabla_s\widetilde{g}_{jl}(\widetilde{g}^{jl} - \widetilde{\xi}^j\widetilde{\xi}^l) \\
&= \sigma\widetilde{g}^{is}[\nabla_l(-\widetilde{\xi}^l\widetilde{\theta}_s) - \nabla_l(\frac{1}{\sigma}(g^{jl} - \xi^j\xi^l))\widetilde{g}_{js}] \\
&\quad + \frac{1}{2}\sigma\widetilde{g}^{is}\widetilde{g}_{jl}\nabla_s[\frac{1}{\sigma}(g^{jl} - \xi^j\xi^l)] \\
&= \sigma\widetilde{g}^{is}\nabla_l(-\widetilde{\xi}^l\widetilde{\theta}_s) - \sigma\nabla_l[\frac{1}{\sigma}(g^{il} - \xi^i\xi^l)] \\
&\quad + \frac{1}{2}\sigma\widetilde{g}^{is}\widetilde{g}_{jl}\nabla_s[\frac{1}{\sigma}(g^{jl} - \xi^j\xi^l)] \\
&= \sigma\widetilde{g}^{is}\nabla_l(-\widetilde{\xi}^l\widetilde{\theta}_s) - \sigma(-\frac{\sigma_l}{\sigma^2})(g^{il} - \xi^i\xi^l) \\
&\quad + \frac{1}{2}\sigma\widetilde{g}^{is}\widetilde{g}_{jl}(\frac{-\sigma_s}{\sigma^2})(g^{jl} - \xi^j\xi^l) - \frac{1}{2}\sigma\widetilde{g}^{is}\widetilde{g}_{jl}\frac{1}{\sigma}\nabla_s(\xi^j\xi^l),
\end{aligned}$$

then we compute

$$\sigma\widetilde{g}^{is}\nabla_l(-\widetilde{\xi}^l\widetilde{\theta}_s) = \frac{n}{\sigma}\xi(\sigma)\widetilde{\xi}^i + \frac{\sigma^j}{\sigma}J^l_j\nabla_l\xi^i + \sigma^j\widetilde{\xi}^i\widetilde{\xi}^s J^l_j\nabla_l\theta_s,$$

$$\frac{\sigma_l}{\sigma}(g^{il} - \xi^i\xi^l) = \frac{\sigma^i}{\sigma} - \frac{1}{\sigma}\xi(\sigma)\xi^i,$$

$$\frac{1}{2}\sigma\widetilde{g}^{is}\widetilde{g}_{jl}(\frac{-\sigma_s}{\sigma^2})(g^{jl} - \xi^j\xi^l) = -\frac{n}{\sigma}\sigma^i + \frac{n}{\sigma}\xi(\sigma)\xi^i - n\widetilde{\xi}^i\widetilde{\xi}^s\sigma_s,$$

$$-\frac{1}{2}\widetilde{g}^{is}\widetilde{g}_{jl}\nabla_s(\xi^j\xi^l) = -\nabla^i\theta_l\frac{\sigma^j}{\sigma}J^l_j - \sigma^j\widetilde{\xi}^i\widetilde{\xi}^s J^l_j\nabla_s\theta_l,$$

at last we have

$$\widetilde{A}^i_{jl}(g^{jl} - \xi^j\xi^l) = \frac{n}{\sigma}\xi(\sigma)\xi^i - \frac{n}{\sigma}\sigma^i.$$

■

Proposition 3.12 *([Tan89]) For a function f on M, we have*

$$\widetilde{\triangle}_P f = \frac{1}{\sigma}\triangle_P f + \frac{n}{\sigma^2}(d\sigma, df)_P. \tag{3.11}$$

Proof.

$$\begin{aligned}
\widetilde{\triangle}_P f &= (\widetilde{g}^{rs} - \widetilde{\xi}^r\widetilde{\xi}^s)\widetilde{\nabla}_r f_s \\
&= \frac{1}{\sigma}(g^{rs} - \xi^r\xi^s)(\nabla_r f_s - \widetilde{A}^a_{rs}f_a) \\
&= \frac{1}{\sigma}\triangle_P f + \frac{n}{\sigma^2}(d\sigma, df)_P.
\end{aligned}$$

■

Chapter 4

The CR Yamabe problem

D. Jerison and J. M. Lee [JL87] considered a Yamabe type problem on CR manifolds. To distinguish it with the Riemannian Yamabe problem, it is called the CR Yamabe problem. It's the following question.

For a given compact, strictly pseudoconvex CR manifold, is it possible to find a choice of pseudohermitian structure with constant Webster scalar curvature?

This problem was solved in affirmative due to D. Jerison, J. M. Lee, N. Gamara and R.Yacoub (see [JL87], [JL89], [GY01] and [Gam01]).

Let (M, θ) be a CR manifold. Under a conformal change $\theta \to \widetilde{\theta} = u^{\frac{2}{n}}\theta$, the transformation law (3.10) gives

$$\widetilde{W} = (-\mu\triangle_P u + Wu)u^{-1-\frac{2}{n}}, \quad \mu = 2(2 + \frac{2}{n}).$$

Therefore the CR Yamabe equation becomes

$$-\mu\triangle_P u + Wu = \lambda u^{1+\frac{2}{n}}.$$

Similar to the Riemannian Yamabe problem, the CR Yamabe invariant $\lambda(M, \theta)$ is central to the solution of the CR Yamabe problem. Its solution contains several parts.

(a) ([JL87]) Let M be a compact, orientable, strictly pseudoconvex, integrable CR manifold of dimension $2n+1$, θ any contact form on M. Then

  (i) $\lambda(M, \theta)$ depends only on the CR structure of M, not the choice of θ;

  (ii) $\lambda(M, \theta) \leq \lambda(S^{2n+1})$, in which $S^{2n+1} \subset \mathbb{C}^{n+1}$ is the sphere with its standard CR structure;

  (iii) If $\lambda(M, \theta) < \lambda(S^{2n+1})$, then the CR Yamabe problem has a solution.

(b) ([JL89]) Suppose M is compact, strictly pseudoconvex, $(2n+1)$-dimensional CR manifold. If $n \geq 2$ and M is not locally CR equivalent to S^{2n+1}, then $\lambda(M, \theta) < \lambda(S^{2n+1})$, and thus the CR Yamabe problem on CR manifold M has a solution.

(c) ([GY01]) Let (M, θ) be a compact (2n+1)-dimensional CR manifold, locally CR equivalent to the sphere S^{2n+1}, then the CR Yamabe problem has a solution on M.

(d) ([Gam01]) Let (M, θ) be a compact CR 3-dimensional manifold, not locally equivalent to the sphere S^3, then the CR Yamabe problem has a solution on M.

4.1 Basic notations

For a CR manifold (M, θ), we have seen in section 3.3 that it has a naturally defined almost complex structure given by

$$J(X + \overline{X}) = i(X - \overline{X}) \quad for \quad X \in T_{1,0}.$$

The Levi form is given by

$$L_\theta(V, W) = \frac{1}{2} < d\theta, V \wedge JW >= d\theta(V, JW).$$

By the partial integrability condition L_θ is compatible with J. Therefore (M, θ, J, g) is a contact metric manifold with $g = L + \theta \otimes \theta$.

For another representative element $\widetilde{\theta} = \sigma\theta$, $\widetilde{J}$ is defined as

$$\widetilde{J} = J + \widetilde{J}(\xi)\theta,$$

and $\widetilde{g}$ is given by

$$\widetilde{g} = d\widetilde{\theta}(\cdot, \widetilde{J}\cdot) + \widetilde{\theta} \otimes \widetilde{\theta}.$$

Now take a conformal pseudohermitian structure represent $\widetilde{\theta} = u^{\frac{2}{n}}\theta$, the Webster scalar curvature transformation law gives

$$\widetilde{W} = (-\mu \triangle_P u + Wu)u^{-1-\frac{2}{n}}, \quad \mu = 2(2 + \frac{2}{n}),$$

according to (3.10).

This formula was first given by J. M. Lee [Lee86] for CR manifolds. C. Fefferman [Fef76a] [Fef76b] constructed a pseudo-Riemannian metric g of Lorentz signature, defined on the total space of a certain circle bundle C over M. In [Fef76a] [Fef76b], M is assumed to be an embedded hypersurface in $\mathbb{C}^{n+1}$, various intrinsic characterizations of g on an abstract CR manifold are known (see [BDS77], [Far86] and [Lee86]). In the following we will give a rough description.

If θ is replaced by $\widetilde{\theta} = u^{p-2}\theta$, then $\widetilde{g} = u^{p-2}g$, where $p = \frac{2(2n+2)}{2n+2-2} = 2+\frac{2}{n}$. Let $\Box$ denote the (Laplace-Beltrami) wave operator of g and R its scalar curvature. By (2.1),

$$\widetilde{R} = (-a\Box u + Ru)u^{1-p},$$

where $a = \frac{4(2n+2-1)}{2n+2-2} = \frac{2(2n+1)}{n}$.

From now on we denote $p = 2 + \frac{2}{n}$.

Since g is invariant under the action of S^1 on C, the operator $\Box$ pushes forward under projection $\pi : C \to M$ to an operator $\pi_*\Box$ on M. Moreover, R is constant on the fibers of C by S^1-invariance, so it projects to a function $\pi_* R$ on M. J. M. Lee [Lee86] showed that $\pi_*\Box = 2\triangle_b$ and $\pi_* R = \frac{2(2n+1)}{n+1}W$, where W is the Webster scalar curvature. Therefore

$$\widetilde{W} = (-\frac{2n+2}{n}\triangle_b u + Wu)u^{1-p}. \tag{4.1}$$

This is why the CR Yamabe problem on CR manifolds is to seek constant Webster scalar curvature. We would like to use notation $\triangle_b$ on CR manifolds due to the history and $\triangle_P$ on general contact metric manifolds.

Thus, the CR Yamabe problem on CR manifolds is led to solve the CR Yamabe equation:

$$-\nu\triangle_b u + Wu = \lambda u^{1+\frac{2}{n}}, \quad \nu = 2 + \frac{2}{n}, \tag{4.2}$$

for some constant λ.

We can define the extremal problem

$$\lambda(M,\theta) = inf\{A(f); \quad B(f) = 1\}, \tag{4.3}$$

where

$$A(f) = \int_M (\nu|df|_b^2 + Wf^2)dv, \quad B(f) = \int_M |f|^p dv.$$

Proposition 4.1 *([JL87]) Let (M,θ) be a compact CR manifold. Then $\lambda(M,\theta)$ is invariant under gauge transformation of contact form.*

Proof. Let $\widetilde{\theta} = \sigma\theta$ and $\widetilde{f} = \frac{f}{u}$, where $\sigma = u^{\frac{2}{n}}$. Therefore

$$\int_M \widetilde{f}^p d\widetilde{v} = \int_M f^p dv.$$

By formula (3.11), we have

$$\begin{aligned}
-\nu\widetilde{\triangle}_b\widetilde{f}+\widetilde{W}\widetilde{f} &= -\nu[\frac{1}{\sigma}\triangle_b(\frac{f}{u})+\frac{n}{\sigma^2}(d\sigma,d(\frac{f}{u}))_b]+\widetilde{W}\frac{f}{u}\\
&= -\nu u^{-1-\frac{2}{n}}\triangle_b f+\nu u^{-2-\frac{2}{n}}f\triangle_b u+2\nu u^{-2-\frac{2}{n}}(df,du)_b\\
&\quad -2\nu u^{-3-\frac{2}{n}}f|du|_b^2-2\nu u^{-2-\frac{2}{n}}(du,df)_b+2\nu u^{-3-\frac{2}{n}}f|du|_b^2\\
&\quad +u^{-1-\frac{2}{n}}(-\nu\triangle_b u+Wu)\frac{f}{u}\\
&= u^{1-p}(-\nu\triangle_b f+Wf).
\end{aligned}$$

Thus,

$$\int_M(-\nu\widetilde{\triangle}_b\widetilde{f}+\widetilde{W}\widetilde{f})\widetilde{f}d\widetilde{v}=\int_M(-\nu\triangle_b f+Wf)fdv,$$

i.e.

$$\int_M(\nu|d\widetilde{f}|_b^2+\widetilde{W}\widetilde{f}^2)d\widetilde{v}=\int_M(\nu|df|_b^2+Wf^2)dv.$$

So $\lambda(M,\theta)$ is invariant under gauge transformation of contact form. ■

We call $\lambda(M,\theta)$ the CR Yamabe invariant. It's central to the CR Yamabe problem.

In [JL87] the answer to the Yamabe problem on CR manifolds is the following main theorem.

Theorem 4.2 *([JL87]) Let M be a compact, orientable, strictly pseudoconvex, integrable CR manifold of dimension $2n+1$, θ any contact form on M.*

(i) $\lambda(M,\theta)$ depends only on the CR structure of M, not the choice of θ.

(ii) $\lambda(M,\theta)\le\lambda(S^{2n+1})$, in which $S^{2n+1}\subset\mathbb{C}^{n+1}$ is the sphere with its standard CR structure.

(iii) If $\lambda(M,\theta)<\lambda(S^{2n+1})$, then the infimum is attained by a positive smooth solution to the CR Yamabe equation (4.2), thus the contact form $\widetilde{\theta}=u^{p-2}\theta$ has constant Webster scalar curvature $W=\lambda(M,\theta)$.

Theorem 4.2(i) is just proposition 4.1. Theorem 4.2(ii) is an analogue of Aubin's Theorem 2.7 in the Riemannian Yamabe problem. To explain Theorem 4.2(ii), we first introduce an important model, i.e. the Heisenberg group H^n.

The Heisenberg group H^n is the Lie group whose underlying manifold is $\mathbb{C}^n\times\mathbb{R}$ with coordinates $(z,t)=(z^1,\cdots,z^n,t)$ and whose group multiply is given by

$$(z,t)(z',t')=(z+z',t+t'+2Imz\cdot\overline{z}'),$$

where $z\cdot\overline{z}'=\sum_{j=1}^n z^j\overline{z'}^j$.

Define a norm in H^n by

$$|x|_{H^n} = |(z,t)|_{H^n} = (|z|^4 + t^2)^{\frac{1}{4}}$$

and dilations by

$$x = (z,t) \mapsto \delta x = (\delta z, \delta^2 t), \quad \delta > 0.$$

The vector fields

$$Z_j = \frac{\partial}{\partial z^j} + i\overline{z}^j \frac{\partial}{\partial t}, \quad j = 1, \cdots, n$$

are invariant with respect to the group multiplication on the left, and homogeneous of degree 1 with respect to the dilations. Then $T_{1,0} = span\{Z_1, \cdots, Z_n\}$ gives a left invariant CR distribution on H^n.

The real 1-form

$$\theta_0 = dt + \sum_{j=1}^{n} (iz^j d\overline{z}^j - i\overline{z}^j dz^j)$$

annihilates $T_{1,0}$, we take it to be the contact form of H^n.

Let

$$L_0 = \sum_{j=1}^{n} (Z_j \overline{Z}_j + \overline{Z}_j Z_j).$$

Then the operator $\triangle_b$ associated to θ_0 is L_0. Since $d\theta_0 = 2i dz^j \wedge d\overline{z}^j$, it follows that $\xi = \frac{\partial}{\partial t}$.

The coframe dual to $\{Z_i\}$ is $\{\theta^i = dz^i\}$. Therefore all the coefficients of the Tanaka connection and torsion vanish, i.e. $\omega_\alpha^\beta = \tau^\gamma = 0$. Hence the Webster scalar curvature of (H^n, θ_0) is identically zero.

The extremal problem (4.3) for H^n is

$$\lambda(H^n, \theta_0) = \inf\{\int_{H^n} \nu |du|^2_{\theta_0} \theta_0 \wedge d\theta_0^n : \int_{H^n} |u|^p \theta_0 \wedge d\theta_0^n = 1\}.$$

The Carley transform is a biholomorphism between the unit ball in $\mathbb{C}^{n+1}$ and the Siegel upper half space $D = \{(z,w) \in \mathbb{C}^n \times \mathbb{C} : Imw > |z|^2\}$, given by

$$w = i(\frac{1-\zeta^{n+1}}{1+\zeta^{n+1}}), \quad z^k = \frac{\zeta^k}{1+\zeta^{n+1}}, \quad k = 1, \cdots, n, \tag{4.4}$$

where $\zeta \in \mathbb{C}^{n+1}, |\zeta| < 1$. When restricted to the boundary, this transformation gives a CR equivalence between $S^{2n+1} \backslash (0, \cdots, 0, -1)$ and ∂D. The Heisenberg group is identified with ∂D by $(z,t) \leftrightarrow (z, t + i|z|^2) = (z,w)$. Denote by F :

$S^{2n+1} \backslash (0, \cdots, 0, -1) \to H^n$ the mapping given by (4.4) and the correspondence $\partial D = H^n$. Explicitly,

$$z^k = F^k(\zeta) = \frac{\zeta^k}{1+\zeta^{n+1}}, \quad t = F^{n+1}(\zeta) = \frac{-i(\zeta^{n+1} - \overline{\zeta}^{n+1})}{|1+\zeta^{n+1}|^2}, \tag{4.5}$$

and

$$\zeta^k = \frac{2z^k}{1 - i(t + i|z|^2)}, \quad \zeta^{n+1} = \frac{1 + i(t + i|z|^2)}{1 - i(t + i|z|^2),} \tag{4.6}$$

$k = 1, 2, \cdots, n$.

Choose the standard contact form for S^{2n+1}

$$\theta_1 = i(\overline{\partial} - \partial)|\zeta|^2 = i \sum_{j=1}^{n+1} (\zeta^j d\overline{\zeta}^j - \overline{\zeta}^j d\zeta^j). \tag{4.7}$$

Then

$$F^* \theta_0 = \frac{1}{|1+\zeta^{n+1}|^2} \theta_1.$$

Therefore by proposition 4.1, for $v(\zeta) = |1+\zeta^{n+1}|^{-n} u \circ F(\zeta)$,

$$\int_{S^{2n+1}} (\nu |dv|^2_{\theta_1} + W_n v^2) \theta_1 \wedge d\theta_1^n = \int_{H^n} \nu |du|^2_{\theta_0} \theta_0 \wedge d\theta_0^n,$$

$$\int_{S^{2n+1}} |v|^p \theta_1 \wedge d\theta_1^n = \int_{H^n} |u|^p \theta_0 \wedge d\theta_0^n,$$

where $W_n = n(n+1)/2$ is the Webster scalar curvature associated to θ_1. Thus the extremal problems for H^n and S^{2n+1} are the same. In particular, $\lambda(H^n, \theta_0) = \lambda(S^{2n+1}, \theta_1)$.

Folland and Stein [FS74] constructed normal coordinates which showed how closely the Heisenberg group approximates a general strictly pseudoconvex CR manifold (M, θ): The exponential map, exp_a for $a \in M$, is a diffeomorphism of a neighborhood U_a of the origin in H^n, onto a neighborhood V_a of a in M, and exp_a^{-1} defines a system of local coordinates on V_a, called normal coordinates.

Theorem 4.2(ii) is an analogue of $\lambda(M, g) \leq \lambda(S^n)$ in the Riemannian Yamabe problem. The way to prove it is also similar. The class of test functions defining $\lambda(H^n, \theta_0)$ can be restricted to C^∞ functions with compact support (see [JL87]). Choose $u \in C_0^\infty(H^n)$ such that $B_{\theta_0}(u) = 1$, $A_{\theta_0}(u) < \lambda(H^n) + \epsilon$. Denote

$$u_\delta(x) = \delta^{-n} u(\delta^{-1} x).$$

Therefore

$$B_{\theta_0}(u_\delta) = B_{\theta_0}(u) = 1, \quad A_{\theta_0}(u_\delta) = A_{\theta_0}(u) < \lambda(H^n) + \epsilon.$$

For $a \in M$, $exp_a^{-1} : V_a \to U_a$ is a normal coordinates of $V_a \subset M$. Define

$$v_\delta(y) = u_\delta(exp_a^{-1} y).$$

For δ sufficiently small, the support of u_δ is contained in $exp_a^{-1}(V_a)$. Thus v_δ has compact support in V_a and can be extended by zero outside V_a to a function in $C^\infty(M)$. Therefore,

$$\lim_{\delta \to 0} B_\theta(v_\delta) = 1, \quad \lim_{\delta \to 0} A_\theta(v_\delta) = A_{\theta_0}(u) \leq \lambda(H^n) + \epsilon.$$

Since ϵ can be arbitrary small positive number, $\lambda(M, \theta) \leq \lambda(S^{2n+1})$. We will continue in next two sections to introduce the analytic preliminaries and 4.2(iii).

4.2 Analytic aspect on CR manifolds

Before going to explain Theorem 4.2(iii), we have to introduce the analysis on CR manifolds. We introduce the Folland-Stein Sobolev space S_k^p and Folland-Stein Hölder space Γ_β in this section. They are suitable spaces for the analysis of the CR Yamabe problem which is related to the sublaplacian $\triangle_b$. Embedding theorems and a priori estimates for the sublaplacian play important roles.

Let U be a relatively compact subset of a normal coordinate neighborhood of M, with contact form θ and pseudohermitian frame $\{Z_1, \cdots, Z_n\}$. Denote by $a \in M$ the origin of U. Let $X_j = ReZ_j$ and $X_{j+n} = ImZ_j$ for $j = 1, \cdots, n$. Denote $X^\alpha = X_{\alpha_1} \cdots X_{\alpha_k}$, where $\alpha = (\alpha_1, \cdots, \alpha_k)$, each α_j is an integer $1 \leq \alpha_j \leq 2n$, and denote $l(\alpha) = k$. Define the norms

$$||f||_{S_k^p(U)} = \sup_{l(\alpha) \leq k} ||X^\alpha f||_{L^p(U)},$$

where

$$||g||_{L^p(U)} = (\int_U |g|^p \theta \wedge d\theta^n)^{1/p}.$$

The Folland-Stein space $S_k^p(U)$ is defined as the completion of $C_0^\infty(U)$ with respect to the norm $|| \cdot ||_{S_k^p(U)}$.

The function

$$\rho(x, y) = |exp_a^{-1}(x) - exp_a^{-1}(y)|_{H^n}$$

is a natural distance function on U. For $0 < \beta < 1$ define

$$\Gamma_\beta(U) = \{f \in C^0(\overline{U}) : |f(x) - f(y)| \leq C\rho(x, y)^\beta\}$$

with norm

$$||f||_{\Gamma_\beta(U)} = \sup_{x \in U} |f(x)| + \sup_{x, y \in U} \frac{|f(x) - f(y)|}{\rho(x, y)^\beta}.$$

For any $k \geq 1$ and $k < \beta < k+1$ define

$$\Gamma_\beta(U) = \{f \in C^0(\overline{U}) : X^\alpha f \in \Gamma_{\beta-k}(U) \quad for \quad l(\alpha) \leq k\}$$

with norm

$$||f||_{\Gamma_\beta(U)} = \sup_{x \in U} |f(x)| + \sup_{x,y \in U, l(\alpha) \leq k} \frac{|X^\alpha f(x) - X^\alpha f(y)|}{\rho(x,y)^{\beta-k}}.$$

Notice that the norms above depend on the choice of pseudo-hermitian frame.

For a compact strictly pseudo-convex pseudo-hermitian manifold M, choose a finite open covering $\{U_1, \cdots, U_m\}$ for which each U_j has the properties of U above. Choose a C^∞ partition of unity ϕ_i subordinate to this covering, and define

$$S_k^p(M) = \{f \in L^1(M) : \phi_j f \in S_k^p(U_j) \quad for \quad all \quad j\},$$
$$\Gamma_\beta(M) = \{f \in C^0(M) : \phi_j f \in \Gamma_\beta(U_j) \quad for \quad all \quad j\}.$$

The standard Hölder space, $\Lambda_\beta(U)$ $0 < \beta < 1$, is defined by

$$\Lambda_\beta(U) = \{f \in C^0(\overline{U}) : |f(x) - f(y)| \leq C||x-y||^\beta\}$$

with norm

$$||f||_{\Lambda_\beta(U)} = sup_{x \in U}|f(x)| + sup_{x,y \in U} \frac{|f(x) - f(y)|}{||x-y||^\beta},$$

where $|| \cdot ||$ is the distance w.r.t. metric $g = L + \theta \otimes \theta$.

For $k < \beta < k+1$, k an integer ≥ 1,

$$\Lambda_\beta(U) = \{f \in C^0(\overline{U}) : (\partial/\partial x)^\alpha f \in \Lambda_{\beta-k}(U) \quad for \quad l(\alpha) \leq k\}$$

with the obvious norm. Then the following estimates are due to Folland and Stein [FS74].

Proposition 4.3 *([FS74]) For each positive non-integer β, each r, $1 < r < \infty$, and each integer $k \geq 1$, there exists a constant C such that for every $f \in C_0^\infty(U)$,*

(a) $||f||_{L^s(U)} \leq ||f||_{S_k^r(U)}$, *where* $1/s = 1/r - k/(2n+2)$ *and* $1 < r < s < \infty$,

(b) If $1 < r < s < \infty$, *and* $1/s > 1/r - 1/(2n+2)$, *then the unit ball in the space* $S_1^r(U)$ *is compact in* $L^s(U)$,

(c) $||f||_{\Gamma_\beta(U)} \leq C||f||_{S_k^r(U)}$, *where* $1/r = (k-\beta)/(2n+2)$,

(d) $||f||_{\Lambda_{\beta/2}(U)} \leq C||f||_{\Gamma_\beta(U)}$,

(e) $||f||_{S_2^r(U)} \leq C(||\triangle_b f||_{L^r(U)} + ||f||_{L^r(U)})$,

(f) $||f||_{\Gamma_{\beta+2}(U)} \leq C(||\triangle_b f||_{\Gamma_\beta(U)} + ||f||_{\Gamma_\beta(U)})$.

The constant C depends only on the frame constants.

Applying a partition of unity, we conclude the estimates in the last proposition hold with U replaced by a compact strictly pseudo-convex CR manifold M.

The following regularity result follows from these estimates just as in [FS74].

Proposition 4.4 *([FS74]) If $u, v \in L^1_{loc}(U)$, and $\triangle_b u = v$ in the distribution sense on U, then for $\eta \in C_0^\infty(U)$ the following hold.*

(a) If $v \in L^r(U), n+1 < r \leq \infty$, then $\eta u \in \Gamma_\beta(U)$ where $\beta = 2 - (2n+2)/r$.

(b) If $v \in S_k^r(U)$, $1 < r < \infty$, $k = 0, 1, 2, \cdots$, then $\eta u \in S_{k+2}^r(U)$.

(c) If $v \in \Gamma_\beta(U)$, β a non-integer > 0, then $\eta u \in \Gamma_{\beta+2}(U)$.

4.3 The solution of the CR Yamabe problem

Since proposition 4.3(a), the CR Yamabe equation is critical for calculus of variations. As in the Riemannian Yamabe problem, to prove Theorem 4.2(iii), the subcritical case is considered first. For a compact strictly pseudo-convex CR manifold (M, θ), consider for each q, $2 \leq q \leq p$, the extremal problem

$$\lambda_q = \inf\{A(u) : \quad u \in S_1^2(M, g), \quad B_q(u) = 1\},$$

in which

$$A(u) = \int_M (\nu |du|_b^2 + W u^2) \theta \wedge d\theta^n, \quad B_q(u) = \int_M |u|^q \theta \wedge d\theta^n.$$

Lemma 4.5 *([JL87]) For $2 \leq q < p$, there exists a positive C^∞ solution u_q to the equation*

$$-\nu \triangle_b u_q + W u_q = \lambda_q u_q^{q-1}$$

satisfying $A(u_q) = \lambda_q$ and $B_q(u_q) = 1$.

However, the CR analogue of (2.5)

$$(\lambda - \epsilon)(\int_M |f|^p dv_g)^{\frac{2}{p}} \leq a \int_M |df|^2 dv_g + C_{M,\epsilon} \int_M |f|^2 dv_g,$$

can not be achieved since the CR analogue of the gradient on the CR manifold M is not compare to that on H^n.

In the case $\lambda(M, \theta) < 0$, by choosing suitable test function, the negativity of the Yamabe invariant gives $||u_q||_{L^r} < C$ for some $r > p$. With this it is shown that $\{u_q\}$ is in fact in $C^k(M)$ for any k. Then one can choose a subsequence of u_q which converges to a smooth limit. Therefore we get a solution of the CR Yamabe problem in the case $\lambda(M, \theta) < 0$.

The case $\lambda(M,\theta) \geq 0$ is the more difficult part of Theorem 4.2(iii). In this case, following Uhlenbeck's idea, Jerison and Lee [JL87] showed the gradient can not blow up, i.e.

$$\sup_M |du_q|_\theta \to \infty$$

as $q \to p$ can not happen. This is proved by contradiction. In fact if $\sup_M |du_q|_\theta \to \infty$ as $q \to p$, they constructed a function $\widetilde{u}$ on H^n satisfies $||\widetilde{u}||_{L^p} = 1$ and

$$\int_{H^n} \nu |d\widetilde{u}|^2_{\theta_0} \theta_0 \wedge d\theta_0^n \leq \lambda(M,\theta) < \lambda(H^n, \theta_0),$$

this contradicts with the definition of $\lambda(H^n, \theta_0)$.

With the gradient bound, the proof of 4.2(iii) is completed (see [JL87]).

(M,θ) is said to be locally **CR equivalent** to (S^{2n+1}, θ_1) if for any $a \in M$ there exists a neighborhood U_a and a map $F : U_a \to S^{2n+1}$ such that $F^*\theta_1 = u^{\frac{2}{n}}\theta$ on U_a for some $u \in C^\infty(M)$. Jerison and J. M. Lee [JL89] proved an analogy of Aubin's result (see Theorem 2.8).

Theorem 4.6 *([JL89]) Suppose M is a compact, strictly pseudoconvex, 2n+1-dimensional CR manifold. If $n \geq 2$ and M is not locally CR equivalent to S^{2n+1}, then $\lambda(M) < \lambda(S^{2n+1})$, and thus the Yamabe problem on CR manifold M can be solved.*

Jerison and Lee [JL89] constructed pseudohermitian normal coordinates and a family of contact forms θ^ϵ which concentrate more and more around one point q. Define the CR Yamabe functional

$$Y(M,\theta) = \frac{\int_M W\theta \wedge d\theta^n}{(\int_M \theta \wedge d\theta^n)^{2/p}}.$$

The asymptotic expression for θ^ϵ is

$$Y(M,\theta^\epsilon) = \begin{cases} \lambda(S^{2n+1})(1 - c(n)|S(q)|^2\epsilon^4) + O(\epsilon^5) & for \quad n \geq 3, \\ \lambda(S^{2n+1})(1 - c(2)|S(q)|^2\epsilon^4 \log\frac{1}{\epsilon}) + O(\epsilon^4) & for \quad n = 2. \end{cases}$$

Here $S(q)$ is the Chern curvature tensor (see [CM74]) of M evaluated at q and $c(n) > 0$. S is identically zero precisely when M is locally CR equivalent to the sphere. So Theorem 4.6 is proved.

For the CR Yamabe problem, the remaining cases are the conformally flat case and the case when $n = 1$. They are solved by N. Gamara and R. Yacoub (see [GY01] and [Gam01]).

Theorem 4.7 *([GY01]) Let (M,θ) be a compact (2n+1)-dimensional CR manifold, locally CR equivalent to the sphere S^{2n+1}, then the equation*

$$-\nu\triangle_b u + W_0 u = u^{p-1} \tag{4.8}$$

has a smooth solution $u > 0$ on M.

Theorem 4.8 *([Gam01]) Let (M,θ) be a compact CR 3-dimensional manifold, not locally equivalent to the sphere S^3, then (4.8) has a smooth solution $u > 0$ on M.*

The proof of these theorems uses the techniques developed by A. Bahri and H. Brezis (see [Bah89] and [BB96]).

4.4 The CR Yamabe solutions on the sphere

In this section we study the CR Yamabe solutions on the standard sphere (S^{2n+1}, θ_1), where θ_1 is given by (4.7). We will use the CR Yamabe solutions to check what happens to the contact Yamabe flow on the standard sphere in section 5.3.

The CR Yamabe problem on S^{2n+1} has many similarities as the Riemannian Yamabe problem on S^n (see section 2.1.4).

Theorem 4.9 *([JL87]) There exists a positive C^∞ contact form $\theta = u^{p-2}\theta_1$ on S^{2n+1} for which the infimum $\lambda(S^{2n+1}, \theta_1)$ is attained.*

Theorem 4.10 *([JL88]) If θ is a contact form associated with the standard CR distribution on the sphere which has constant Webster scalar curvature, then θ is obtained from a constant multiple of the standard form θ_1 by a CR automorphism of the sphere.*

Corollary 4.11 *([JL88]) The best constant in the Sobolev inequality*

$$(\int_{H^n} |u|^p \theta \wedge d\theta^n)^{\frac{2}{p}} \leq C \int_{H^n} \sum_{\alpha=1}^{n} (|Z_\alpha u|^2 + |\overline{Z}_\alpha u|^2)\theta \wedge d\theta^n$$

is $C = \frac{1}{2\pi n^2}$. Equality is attained only by the functions

$$u_1(z,t) = K|w + z \cdot \mu + \lambda|^{-n},$$

where $K, \lambda \in \mathbb{C}, w = t + i|z|^2, Im\lambda > |\mu|^2/4$ and $\mu \in \mathbb{C}^n$. These solutions are obtained from the function $K|w+i|^{-n}$ by left translations and dilations $(z,t) \mapsto (\delta z, \delta^2 t)$ on the Heisenberg group.

In fact the solutions $\{u_1\}$ are obtained from dilation of $(z,t) \to (\delta z, \delta^2 t)$ and left translation given by $(\frac{i}{2}\overline{\mu}, \lambda - i(\frac{|\mu|^2}{4} + \frac{1}{\delta^2}))$.

Since proposition 4.1 and $F^*\theta_0 = \frac{1}{|1+\zeta^{n+1}|^2}\theta_1$, the solutions u such that $u^{\frac{2}{n}}\theta_1$ have constant Webster scalar curvature are

$$\begin{aligned} u(\zeta) &= K|i(\frac{1-\zeta^{n+1}}{1+\zeta^{n+1}}) + \frac{\mu^k\zeta^k}{1+\zeta^{n+1}} + \lambda|^{-n}|1+\zeta^{n+1}|^{-n} \\ &= K|i(1-\zeta^{n+1}) + \lambda(1+\zeta^{n+1}) + \sum_{k=1}^{n} \mu^k \zeta^k|^{-n}. \end{aligned} \tag{4.9}$$

Thus for $u(\zeta) = K|i(1-\zeta^{n+1}) + \lambda(1+\zeta^{n+1}) + \sum_{k=1}^{n} \mu^k \zeta^k|^{-n}$,

$$-\nu \triangle_b^{\theta_1} u + W(\theta_1) u = c u^{p-1} \quad \textit{and} \quad W(u^{\frac{2}{n}} \theta_1) = c.$$

Chapter 5

The Contact Yamabe flow

First let us recall the contact Yamabe problem. Given a contact distribution or equivalently a conformal class of contact forms, one can assign an associated metric together with an almost complex structure J to it. Tanno [Tan89] generalized the Tanaka connection and the Webster scalar curvature defined on CR manifolds onto any contact metric manifold (M, θ_0, J, g_0). The contact Yamabe problem is to find a choice in the conformal class $[\theta_0]$ with J fixed on the contact distribution such that its Webster scalar curvature is constant. If we write $\theta = u^{\frac{2}{n}}\theta_0$, then

$$W = (-4(1+\frac{1}{n})\triangle^0_P u + W_0 u)u^{-1-\frac{2}{n}}.$$

Therefore the contact Yamabe problem is to solve the contact Yamabe equation

$$-4(1+\frac{1}{n})\triangle^0_P u + W_0 u = \lambda u^{1+\frac{2}{n}},$$

for some constant λ. In this chapter we use the contact Yamabe flow to deal with the contact Yamabe problem. The contact Yamabe flow defined on the conformal class θ_0 is given by

$$\begin{cases} \frac{\partial\theta(x,t)}{\partial t} = (\overline{W}(t) - W(x,t))\theta(x,t) \\ \theta(x,0) = \theta^0 \in [\theta_0]), \end{cases} \tag{5.1}$$

where W denotes the Webster scalar curvature with respect to the contact form θ and $\overline{W}$ denotes its average $\overline{W} = \frac{\int_M W\theta\wedge d\theta^n}{\int_M \theta\wedge d\theta^n}$. Along the contact Yamabe flow (5.1) J is fixed in the following sense:

For each point $x \in M$ the action of J_0 and $J(t)$ are identical on the contact distribution and $J(t)\xi(t) = 0$.

The contact Yamabe flow (5.1) has been considered by Shucheng Chang and JihHsin Cheng [CC02]. A stationary solution of this flow provides a desired contact form with constant Webster scalar curvature.

This chapter is devoted to prove Theorem 1.1 and 1.2.

Theorem 1.1 Let (M, θ_0, J, g_0) be a connected, compact contact Riemannian manifold of dimension $2n+1$.

(a) The contact Yamabe flow (5.1) admits a smooth solution on a maximal time interval $[0, T), 0 < T \leq \infty$.

(b) If the contact Yamabe invariant $\lambda(M, [\theta_0])$ is negative, then there exists a contact metric structure $(M, \theta_\infty, J, g_\infty)$ with negative constant Webster scalar curvature. In particular, for any choice $\theta^0 \in [\theta_0]$ satisfying $W(\theta^0) < 0$ the solution $\theta(t)$ of (5.1) exists for all time and $\theta(t)$ converges in C^0 norm to a smooth contact form $\theta_\infty \in [\theta_0]$ in C^0 norm which has negative constant Webster scalar curvature.

Theorem 1.2 Let (M, θ_0, J, g_0) be a K-contact metric manifold. Then the contact metric Yamabe flow (5.1) with initial data $\theta^0 = \theta_0$ exists for all time and converges smoothly to a smooth limit θ_∞ with constant Webster scalar curvature.

5.1 Standard results for the contact Yamabe flow

In this section we reformulate the contact Yamabe flow (5.1) as a heat equation and prove flow (5.1) has short time existence. Some necessary evolution equations are also computed in this section.

5.1.1 Basic materials

We have seen in proposition 3.10 that if we write $\theta = u^{\frac{2}{n}}\theta_0$, then

$$W = (-\mu \triangle_P^0 u + W_0 u)u^{1-p}, \tag{5.2}$$

where $\triangle_P^0$ denotes the sublaplacian w.r.t. the contact form θ_0, W_0 is the Webster scalar curvature of θ_0, $\mu = 2(2 + \frac{2}{n})$ and $p = 2 + \frac{2}{n}$. The contact Yamabe flow (5.1) can be written as

$$\frac{\partial}{\partial t}(u^{\frac{2}{n}}\theta_0) = [\overline{W} - (-\mu \triangle_P^0 u + W_0 u)u^{-1-\frac{2}{n}}]u^{\frac{2}{n}}\theta_0.$$

Therefore we have

$$\frac{\partial u}{\partial t} = \frac{n}{2}\mu(u^{-\frac{2}{n}}\triangle_P^0 u - \frac{1}{\mu}W_0 u^{1-\frac{2}{n}} + \frac{1}{\mu}\overline{W}u). \tag{5.3}$$

Note that sometimes we omit the factor $\frac{n}{2}\mu$ in (5.3).

Flow (5.1) is also equivalent to

$$\frac{\partial u}{\partial t} = \frac{n}{2}(\overline{W} - W)u. \tag{5.4}$$

Denote the contact Yamabe energy of u by

$$A(u) = \int_M (\mu |du|_P^2 + W_0 u^2)\theta_0 \wedge d\theta_0^n,$$

where $|\cdot|_P$ is the sub-inner product w.r.t. θ_0. For $\theta = u^{\frac{2}{n}}\theta_0$, we denote $A(\theta) = A(u)$. Therefore by (5.2)

$$A(\theta) = \int_M W\theta \wedge d\theta^n.$$

To solve the CR Yamabe problem on CR manifolds by an elliptic approach the extremal problem (4.3) has been defined where we try to minimize $A(u)$ under the condition $B(u) = 1$. Corresponding to the flow approach, we have the following.

Proposition 5.1 *Along the contact Yamabe flow (5.1) it satisfies:*

(i) The volume is preserved, w.l.o.g. we assume $V \equiv 1$;

(ii) The contact Yamabe energy A is nonincreasing since

$$\frac{d}{dt}\int_M W dv = -n\int_M (\overline{W} - W)^2 dv. \tag{5.5}$$

Proof. (i)

$$\begin{aligned}
\frac{dV}{dt} &= \frac{d}{dt}\int_M \theta \wedge d\theta^n \\
&= \int_M \frac{\partial \theta}{\partial t} \wedge d\theta^n + n\int_M \theta \wedge d(\frac{\partial \theta}{\partial t}) \wedge d\theta^{n-1} \\
&= (n+1)\int_M (\overline{W} - W)\theta \wedge d\theta^n = 0.
\end{aligned}$$

(ii)

$$\begin{aligned}
A(u) &= \int_M (\mu |du|_P^2 + W_0 u^2)\theta_0 \wedge d\theta_0^n \\
&= \int_M (-\mu u \triangle_P^0 u + W_0 u^2) dv_0.
\end{aligned}$$

By (5.2), (5.4) and the identity (3.3):

$$\int_M f \triangle_P^0 g dv_0 = \int_M g \triangle_P^0 f dv_0,$$

we have

$$
\begin{aligned}
\frac{d}{dt}\int_M W dv &= \frac{d}{dt}\int_M (-\mu u \triangle_P^0 u + W_0 u^2) dv_0 \\
&= \frac{n}{2}\int_M [-\mu \triangle_P^0 u(\overline{W} - W)u - \mu u \triangle_P^0((\overline{W} - W)u)] dv_0 \\
&\quad + \frac{n}{2}\int_M 2W_0 u^2(\overline{W} - W) dv_0 \\
&= n\int_M [-\mu u \triangle_P^0 u(\overline{W} - W) + u(\overline{W} - W)(W u^{1+\frac{2}{n}} + \mu \triangle_P^0 u)] dv_0 \\
&= n\int_M W(\overline{W} - W) u^{2+\frac{2}{n}} dv_0 \\
&= -n\int_M (\overline{W} - W)^2 dv.
\end{aligned}
$$

■

Proposition 5.2 *Under the contact Yamabe flow (5.1), the evolution equation for the Webster scalar curvature is*

$$
\frac{\partial W}{\partial t} = \frac{n\mu}{2}\triangle_P^\theta W + W(W - \overline{W}). \tag{5.6}
$$

Proof.

$$
\begin{aligned}
\frac{\partial W}{\partial t} &= \frac{\partial}{\partial t}[u^{-1-\frac{2}{n}}(-\mu \triangle_P^0 u + W_0 u)] \\
&= \mu(1 + \frac{2}{n}) u^{-2-\frac{2}{n}} \triangle_P^0 u \frac{\partial u}{\partial t} - \frac{2}{n} W_0 u^{-1-\frac{2}{n}} \frac{\partial u}{\partial t} - \mu u^{-1-\frac{2}{n}} \triangle_P^0 (\frac{\partial u}{\partial t}) \\
&= \mu(1 + \frac{2}{n}) \triangle_P^0 u \frac{n}{2}(\overline{W} - W) u^{-1-\frac{2}{n}} - W_0(\overline{W} - W) u^{-\frac{2}{n}} \\
&\quad - \frac{n\mu}{2} u^{-1-\frac{2}{n}} \triangle_P^0[(\overline{W} - W)u] \\
&= \mu(\frac{n}{2} + 1)(\overline{W} - W) u^{-1-\frac{2}{n}} \triangle_P^0 u - (\overline{W} - W)(W + \mu u^{-1-\frac{2}{n}} \triangle_P^0 u) \\
&\quad + \frac{n\mu}{2} u^{-\frac{2}{n}} \triangle_P^0 W - \frac{n\mu}{2}(\overline{W} - W) u^{-1-\frac{2}{n}} \triangle_P^0 u \\
&\quad + n\mu < dW, du >_P u^{-1-\frac{2}{n}} \\
&= -(\overline{W} - W)W + \frac{n\mu}{2} u^{-\frac{2}{n}} \triangle_P^0 W + n\mu < dW, du >_P u^{-1-\frac{2}{n}} \\
&= -(\overline{W} - W)W + \frac{n\mu}{2}\triangle_P^\theta W,
\end{aligned}
$$

where we have used identity (3.11):

$$
\widetilde{\triangle}_P f = \frac{1}{\sigma}\triangle_P f + \frac{n}{\sigma^2}(d\sigma, df)_P,
$$

for $\widetilde{\theta} = \sigma\theta$. Therefore

$$\begin{aligned}\triangle_P^\theta W &= u^{-\frac{2}{n}}\triangle_P^0 W + nu^{-\frac{4}{n}} < du^{\frac{2}{n}}, dW >_P \\ &= u^{-\frac{2}{n}}\triangle_P^0 W + 2u^{-1-\frac{2}{n}} < dW, du >_P .\end{aligned}$$

■

From (5.6) we can also get the evolution equation for the contact Yamabe energy $A(\theta)$ in proposition 5.1(ii). In fact,

$$\begin{aligned}\frac{d}{dt}A(\theta) &= \frac{d}{dt}\int_M W dv \\ &= \int_M [\frac{n\mu}{2}\triangle_P^\theta W + W(W - \overline{W}) + (n+1)(\overline{W} - W)W]dv \\ &= -n\int_M (\overline{W} - W)^2 dv.\end{aligned}$$

5.1.2 The short time existence

In this section we prove the short time existence of the contact Yamabe flow (5.1), i.e. Theorem 1.1(a).

We will prove this by considering the heat equation of u, i.e. (5.3):

$$\frac{\partial u}{\partial t} = \frac{n}{2}\mu(u^{-\frac{2}{n}}\triangle_P^0 u - \frac{1}{\mu}W_0 u^{1-\frac{2}{n}} + \frac{1}{\mu}\overline{W}u).$$

Because the differential operator $\triangle_P^0$ is degenerate, one has to check this heat equation does have a short time solution. Not necessary but for simplicity, we get rid of the term involving $\overline{W}$ by scaling.

Lemma 5.3 *Up to scales on the contact form θ and time t, i.e.*

$$\theta(t) = \psi(\tau)\widetilde{\theta}(\tau) \quad \textit{and} \quad t = \int_0^\tau \psi(s)ds,$$

$$\frac{\partial\widetilde{\theta}}{\partial\tau} = -\widetilde{W}\widetilde{\theta} \quad \textit{and} \quad \frac{\partial\theta}{\partial t} = (\overline{W} - W)\theta$$

are equivalent.

Proof. Assume $\frac{\partial\widetilde{\theta}}{\partial\tau} = -\widetilde{W}\widetilde{\theta}$ and let $\theta(t) = \psi(\tau)\widetilde{\theta}(\tau)$. From proposition 5.1(i) we have

$$0 = \frac{d}{d\tau}\int_M \theta\wedge d\theta^n = \frac{d}{d\tau}\int_M \psi(\tau)^{n+1}\widetilde{\theta}\wedge d\widetilde{\theta}^n.$$

It implies

$$\frac{\psi'}{\psi} = \frac{\int_M \widetilde{W}\widetilde{\theta}\wedge d\widetilde{\theta}^n}{\int_M \widetilde{\theta}\wedge d\widetilde{\theta}^n} = \overline{\widetilde{W}}.$$

Setting $t = \int_0^\tau \psi(s)ds$, it implies $\frac{\partial t}{\partial \tau} = \psi(\tau)$. Thus we have

$$\begin{aligned} \frac{\partial \theta}{\partial t} &= \frac{\partial \theta}{\partial \tau}\frac{\partial \tau}{\partial t} = (\frac{\partial \psi(\tau)}{\partial \tau}\widetilde{\theta} + \psi\frac{\partial \widetilde{\theta}}{\partial \tau})\frac{\partial \tau}{\partial t} \\ &= (\psi\overline{\widetilde{W}}\widetilde{\theta} - \psi\widetilde{W}\widetilde{\theta})\frac{1}{\psi} = (\overline{\widetilde{W}} - \widetilde{W})\widetilde{\theta} \\ &= (\overline{W} - W)\theta. \end{aligned}$$

This proves both equations are equivalent. ∎

Proof of Theorem 1.1(a). By the last lemma we need only to prove the short time existence for the un-normalized flow $\frac{\partial \theta}{\partial t} = -W\theta$. Setting again $\theta(t) = u^{\frac{2}{n}}\theta_0$, we have the following heat equation of $u(x,t)$:

$$\frac{\partial u}{\partial t} = \frac{n}{2}\mu(\triangle_P^0 u - \frac{1}{\mu}W_0 u)u^{-\frac{2}{n}}. \tag{5.7}$$

Let $\epsilon > 0$ be a small constant. We will prove the short time existence by adding term $\epsilon\triangle^0 u$ on the right side then achieving a uniform bound on parameter ϵ for all derivatives of u.

By a constant scale on the time variable, we consider the heat equation

$$\frac{\partial u}{\partial t} = u^{-\frac{2}{n}}(\triangle_P^0 u + \epsilon\triangle^0 u - \frac{1}{\mu}W_0 u). \tag{5.8}$$

Denote its solution as u_ϵ.

The standard parabolic theory tells (5.8) has a solution on a time interval $[0, T_\epsilon]$. We are going to prove that on some interval $[0, T]$ all the derivatives $|\nabla^k u_\epsilon|_{g_{\theta_0}}$ $(k \geq 0)$ are bounded uniformly as $\epsilon \to 0$, then by Arzela-Ascoli theorem we get a solution u of (5.7) on $[0, T]$. Note that all the norms $|\nabla^k u_\epsilon|_{g_{\theta_0}}$ are with respect to g_{θ_0}. u_ϵ is simply written as u below.

Write

$$\triangle_P^0 u = (g^{ij} - \xi^i\xi^j)\nabla_i\nabla_j u := \gamma^{ij}\nabla_i\nabla_j u \quad and \quad f = \frac{1}{\mu}W_0.$$

First, we prove that on some interval [0,T] u is bounded from above and away from zero. Set $u_{max}(t) = \sup_{y\in M} u(y,t)$, $u_{min}(t) = \inf_{y\in M} u(y,t)$. By the maximum principle it follows from (5.8) that

$$\frac{du_{min}(t)}{dt} \geq -cu_{min}^{1-\frac{2}{n}}(t),$$

for some constant $c > 0$. Therefore,

$$u_{min}^{\frac{2}{n}}(t) \geq u_{min}^{\frac{2}{n}}(0) - ct.$$

Thus there exist constants $b > 0$ and $T > 0$ such that $u \geq b$ for $t \in [0, T]$. Also we have

$$\frac{du_{max}(t)}{dt} \leq c u_{max}^{1-\frac{2}{n}}(t),$$

for some constant $c > 0$. Therefore,

$$u_{max}^{\frac{2}{n}}(t) \leq u_{max}^{\frac{2}{n}}(0) + ct.$$

Thus on the interval $[0, T]$,

$$0 < b \leq u \leq B < \infty. \tag{5.9}$$

Second, We prove that on the interval $[0, T]$ $|\nabla u|_{g_{\theta_0}}$ is bounded. Choose normal coordinates and compute

$$\begin{aligned}
&\frac{\partial |\nabla u|^2}{\partial t} = 2\nabla_k u \nabla_k (\frac{\partial u}{\partial t}) \\
= \;& 2\nabla_k u \nabla_k [(\gamma^{ij}\nabla_i \nabla_j u - fu + \epsilon \triangle u) u^{-\frac{2}{n}}] \\
= \;& 2\nabla_k u u^{-\frac{2}{n}} \nabla_k (\gamma^{ij}\nabla_i\nabla_j u - fu + \epsilon\triangle u) \\
& + 2\nabla_k u \nabla_k u (-\frac{2}{n} u^{-1-\frac{2}{n}})(\gamma^{ij}\nabla_i\nabla_j u - fu + \epsilon \triangle u) \\
= \;& 2u^{-\frac{2}{n}}\nabla_k u \nabla_k \gamma^{ij}\nabla_i\nabla_j u + 2u^{-\frac{2}{n}}\gamma^{ij}\nabla_k u \nabla_k \nabla_i \nabla_j u \\
& - 2u^{1-\frac{2}{n}}\nabla_k f \nabla_k u - 2fu^{-\frac{2}{n}}|\nabla u|^2 + 2\epsilon u^{-\frac{2}{n}}\nabla_k u \nabla_k(\triangle u) \\
& - \frac{4}{n} u^{-1-\frac{2}{n}} |\nabla u|^2 (\gamma^{ij}\nabla_i\nabla_j u - fu + \epsilon\triangle u).
\end{aligned}$$

By the Ricci formula

$$\nabla_k \nabla_i \nabla_j u = \nabla_i \nabla_j \nabla_k u - R_{kilj}\nabla_l u$$

and

$$\gamma^{ij}\nabla_i\nabla_j |\nabla u|^2 = 2\gamma^{ij}\nabla_k u \nabla_i \nabla_j \nabla_k u + 2\gamma^{ij}\nabla_i\nabla_k u \nabla_j \nabla_k u,$$

we have:

$$\begin{aligned}
\gamma^{ij}\nabla_k u \nabla_k \nabla_i \nabla_j u &= \gamma^{ij}\nabla_k u (\nabla_i\nabla_j\nabla_k u - R_{kilj}\nabla_l u) \\
&= \frac{1}{2}\triangle_P |\nabla u|^2 - \gamma^{ij}\nabla_i\nabla_k u \nabla_j \nabla_k u - \gamma^{ij} R_{kilj}\nabla_k u \nabla_l u.
\end{aligned}$$

Similarly, by Ricci formula

$$\nabla_k \nabla_i \nabla_i u = \nabla_i \nabla_i \nabla_k u - R_{kili}\nabla_l u = \nabla_i\nabla_i\nabla_k u - R_{kl}\nabla_l u$$

and

$$\triangle |\nabla u|^2 = 2\nabla_i\nabla_i\nabla_k u \nabla_k u + 2|\nabla^2 u|^2,$$

we have

$$\nabla_k u \nabla_k(\triangle u) = \frac{1}{2}\triangle|\nabla u|^2 - |\nabla^2 u|^2 - R_{kl}\nabla_k u \nabla_l u.$$

Thus we have

$$\begin{aligned}
\frac{\partial |\nabla u|^2}{\partial t} &= 2u^{-\frac{2}{n}}\nabla_k \gamma^{ij}\nabla_k u \nabla_i \nabla_j u \\
&\quad +2u^{-\frac{2}{n}}(\frac{1}{2}\triangle_P|\nabla u|^2 - \gamma^{ij}\nabla_i\nabla_k u \nabla_j \nabla_k u - \gamma^{ij}R_{kilj}\nabla_k u \nabla_l u) \\
&\quad -2u^{1-\frac{2}{n}}\nabla_k f \nabla_k u - 2fu^{-\frac{2}{n}}|\nabla u|^2 \\
&\quad +2\epsilon u^{-\frac{2}{n}}(\frac{1}{2}\triangle|\nabla u|^2 - |\nabla^2 u|^2 - R_{kl}\nabla_k u \nabla_l u) \\
&\quad -\frac{4}{n}u^{-1-\frac{2}{n}}|\nabla u|^2\gamma^{ij}\nabla_i\nabla_j u + \frac{4}{n}fu^{-\frac{2}{n}}|\nabla u|^2 \\
&\quad -\frac{4\epsilon}{n}u^{-1-\frac{2}{n}}|\nabla u|^2\triangle u. \qquad (5.10)
\end{aligned}$$

Assume at $x \in M$, $|\nabla u|^2(x,t) = \sup_{y\in M}|\nabla u|^2(y,t)$. It implies $\triangle_P|\nabla u|^2(x,t) \leq 0$ and $\triangle|\nabla u|^2(x,t) \leq 0$. Thus (5.9) and (5.10) imply that for $t \in [0,T]$,

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla u|^2(x,t) &\leq -c_1\gamma^{ij}\nabla_i\nabla_k u \nabla_j\nabla_k u - c_2\epsilon|\nabla^2 u|^2 \\
&\quad +c_3|\nabla_k\gamma^{ij}\nabla_k u \nabla_i\nabla_j u| + c_4 + c_5|\nabla u|^2 \\
&\quad +c_6|\nabla u|^2|\gamma^{ij}\nabla_i\nabla_j u| + c_7\epsilon|\nabla u|^2|\triangle u|. \qquad (5.11)
\end{aligned}$$

Note that all the constants c_j are independent of ϵ.

To continue the computation, we look at some terms of (5.11) more explicitly. In the normal coordinates about x, $\gamma^{ij}(x) = \delta^{ij} - \xi^i\xi^j$. After an orthonormal transformation we can assume $\xi^i(x) = \delta_{1i}$. Thus we have

$$\gamma^{11}(x) = 0,$$

$$\nabla_k\xi^1(x) = \xi^1\nabla_k\xi^1(x) = \frac{1}{2}\nabla_k|\xi|^2(x) = 0$$

and

$$\nabla_k\gamma^{11}(x) = 0.$$

Observe that $|\triangle u|^2 \leq (2n+1)|\nabla^2 u|^2$. It follows from (5.11) that

$$\frac{\partial}{\partial t}|\nabla u|^2(x,t) \leq c_1 + c_2|\nabla u|^4(x,t). \qquad (5.12)$$

Let $h(t) = \sup_{y\in M}|\nabla u|^2(y,t)$, we have

$$\frac{dh}{dt} \leq c(1+h)^2$$

for some constant $c > 0$. Therefore we get

$$|\nabla u|^2 \leq c, \qquad (5.13)$$

on the interval $[0,T]$.

Third, we prove that all the higher order derivatives $|\nabla^k u|_{g_{\theta_0}}$, $k \geq 2$, are bounded. Assume at $x \in M$, $|\nabla^2 u|(x,t) = \sup_{y\in M} |\nabla^2 u|(y,t)$. With estimates (5.9) and (5.13), we have

$$
\begin{aligned}
& \frac{\partial}{\partial t}|\nabla^2 u|^2(x,t) = 2\nabla_k\nabla_l u\nabla_k\nabla_l(\frac{\partial u}{\partial t}) \\
= \quad & 2\nabla_k\nabla_l u\nabla_k\nabla_l[(\gamma^{ij}\nabla_i\nabla_j u - fu + \epsilon\triangle u)u^{-\frac{2}{n}}] \\
= \quad & 2\nabla_k\nabla_l u\nabla_k\nabla_l(u^{-\frac{2}{n}})(\gamma^{ij}\nabla_i\nabla_j u - fu + \epsilon\triangle u) \\
& +2\nabla_k\nabla_l u 2\nabla_k(u^{-\frac{2}{n}})\nabla_l(\gamma^{ij}\nabla_i\nabla_j u - fu + \epsilon\triangle u) \\
& +2\nabla_k\nabla_l u u^{-\frac{2}{n}}\nabla_k\nabla_l(\gamma^{ij}\nabla_i\nabla_j u - fu + \epsilon\triangle u) \\
\leq \quad & c_1 + c_2|\nabla^2 u|^3(x,t) \\
& +4\nabla_k\nabla_l u\nabla_k(u^{-\frac{2}{n}})\nabla_l(\gamma^{ij}\nabla_i\nabla_j u + \epsilon\nabla_i\nabla_i u) \\
& +4u^{-\frac{2}{n}}\nabla_k\nabla_l u\nabla_k\gamma^{ij}\nabla_l\nabla_i\nabla_j u \\
& +2u^{-\frac{2}{n}}\nabla_k\nabla_l u(\gamma^{ij}\nabla_i\nabla_j\nabla_k\nabla_l u + \epsilon\triangle\nabla_k\nabla_l u),
\end{aligned}
$$

thus we have

$$
\begin{aligned}
& \frac{\partial}{\partial t}|\nabla^2 u|^2(x,t) \\
\leq \quad & c_1 + c_2|\nabla^2 u|^3(x,t) \\
& +4\nabla_k\nabla_l u\nabla_k(u^{-\frac{2}{n}})\nabla_l(\gamma^{ij}\nabla_i\nabla_j u + \epsilon\nabla_i\nabla_i u) \\
& +4u^{-\frac{2}{n}}\nabla_k\nabla_l u\nabla_k\gamma^{ij}\nabla_l\nabla_i\nabla_j u \\
& +2u^{-\frac{2}{n}}[\frac{1}{2}\triangle_P(|\nabla^2 u|^2) - \gamma^{ij}\nabla_i\nabla_k\nabla_l u\nabla_j\nabla_k\nabla_l u] \\
& +2\epsilon u^{-\frac{2}{n}}[\frac{1}{2}\triangle(|\nabla^2 u|^2) - |\nabla^3 u|^2] \\
\leq \quad & c_1 + c_2|\nabla^2 u|^3(x,t).
\end{aligned}
$$

With this estimate, we see that $|\nabla^2 u|$ is bounded on the interval $[0,T]$.

Let $\nabla^k u$ denote the derivatives of k-th order, $k \geq 3$. Assume at $x \in M$, $|\nabla^k u|(x,t) = \sup_{y\in M} |\nabla^k u|(y,t)$. We proceed by induction. Assume $|\nabla^s u|_{g_{\theta_0}}$, $0 \leq s \leq k-1$, are bounded on $[0,T]$, one can see

$$
\begin{aligned}
\frac{\partial}{\partial t}|\nabla^k u|^2(x,t) \quad = \quad & 2\nabla^k u\nabla^k[u^{-\frac{2}{n}}(\gamma^{ij}\nabla_i\nabla_j u - fu + \epsilon\triangle u)] \\
\leq \quad & c_1 + c_2|\nabla^k u|^2(x,t) \\
& +c_3|\nabla^k u\nabla^{k-1}(\gamma^{ij}\nabla_i\nabla_j u - fu + \epsilon\triangle u)| \\
& +2\nabla^k u u^{-\frac{2}{n}}\nabla^k(\gamma^{ij}\nabla_i\nabla_j u - fu + \epsilon\triangle u),
\end{aligned}
$$

thus we have

$$
\begin{aligned}
\frac{\partial}{\partial t}|\nabla^k u|^2(x,t) \quad \leq \quad & c_1 + c_2|\nabla^k u|^2(x,t) \\
& + c_3|\nabla^k u \gamma^{ij}\nabla^{k-1}\nabla_i\nabla_j u| + c_4|\nabla^k u \nabla\gamma^{ij}\nabla^{k-1}\nabla_i\nabla_j u| \\
& + c_5\epsilon|\nabla^k u\nabla^{k-1}\triangle u| \\
& + 2u^{-\frac{2}{n}}(\frac{1}{2}\triangle_P|\nabla^k u|^2 - \gamma^{ij}\nabla_i\nabla^k u\nabla_j\nabla^k u) \\
& + c_6|\nabla^k u\nabla\gamma^{ij}\nabla^{k-1}\nabla_i\nabla_j u| \\
& + 2\epsilon u^{-\frac{2}{n}}(\frac{1}{2}\triangle|\nabla^k u|^2 - |\nabla^{k+1}u|^2) \\
\leq \quad & c_1 + c_2|\nabla^k u|^2(x,t).
\end{aligned}
$$

So we finish the proof of short time existence with estimates

$$|\nabla^k u|^2_{max}(t) \leq (1 + |\nabla^k u|^2_{max}(0))e^{ct},$$

for $k \geq 3$. ■

5.2 The contact Yamabe flow with $\lambda(M, \theta_0) < 0$

This section is devoted to prove Theorem 1.1(b). Comparing to Ye's Theorem 2.22, we have to assume an additional condition on the initial data $W(\theta^0) < 0$. For the Riemannian Yamabe flow C^0 bound is enough to imply bounds of all higher order derivatives, due to L^p and Schauder estimates for parabolic equation. However, in the author's knowledge, the absence of the corresponding L^p and Schauder estimates for the heat equation (5.3) makes difficulty. We have to achieve bounds of gradient and higher order derivatives by applying the boundedness of the Webster scalar curvature W. This is the reason we assume the initial data $W(\theta^0) < 0$.

We can define the contact Yamabe invariant $\lambda(M, \theta_0)$ as a natural generalization of (4.3), i.e.

$$\lambda(M, \theta) = inf\{A(f); \quad B(f) = 1\},$$

where

$$A(f) = \int_M (\mu|df|^2_P + Wf^2)dv_0, \quad B(f) = \int_M |f|^p dv_0.$$

Assume $\lambda(M, \theta_0) < 0$. Let u be the first eigenfunction of the conformal sublaplacian $\Box = -\mu\triangle^0_P + W_0$ on M with eigenvalue

$$\lambda_1 = \inf_{u>0} \frac{\int_M (\mu|du|^2_P + W_0 u^2)dv_0}{\int_M u^2 dv_0}.$$

Therefore $\Box u = \lambda_1 u$. λ_1 has the same sign as $\lambda(M, \theta_0)$. Denote $\theta = u^{\frac{2}{n}}\theta_0$ and $W = W(\theta)$. Since $-\mu\triangle^0_P u + W_0 u = \lambda_1 u = Wu^{p-1}$, it follows $W = \lambda_1 u^{2-p} < 0$.

Therefore under the condition $\lambda(M, \theta_0) < 0$, $W(u^{\frac{2}{n}}\theta_0) < 0$. If it's necessary we pass to another conformal element, we assume $W(\theta_0) < 0$. Denote by W_0 the Webster scalar curvature w.r.t. θ_0 and by $W(0)$ the Webster scalar curvature w.r.t. $W(\theta(0))$.

5.2.1 The long-time existence

Having the short time existence, we try to prove the long-time existence. For the short time existence, we don't need any assumption on the initial Webster scalar curvature. But for the long-time existence and convergence, our argument depends on the maximum principle which requires the negativity of the initial Webster scalar curvature.

To prove the long-time existence, we need norm estimates as what we did in proving the short time existence. All the norms are with respect to g_{θ_0}. However straightforward computation as in the proof of the short time existence is not enough now, we still need an observation on the geometric quantity W to guarantee the norm estimates.

The procedure to prove the long-time existence is standard. Assume T^* is the maximal time such that u has a smooth and positive solution on $[0, T^*)$. We will show T^* can't be finite by contradiction. That's, with the assumption $T^* < \infty$ we can prove all derivatives of u are uniformly bounded on the interval $[0, T^*)$. Thus the solution of (5.1) can go beyond T^*, it contradicts the assumption that $[0, T^*)$ is the maximal interval.

At first we prove the C^0 norm is uniformly bounded for the contact Yamabe flow (5.1) under the assumption $W_0 < 0$.

Lemma 5.4 *Assume $W_0 < 0$, then the solution u is uniformly bounded from above and away from zero, i.e.*

$$0 < b \leq u \leq B < \infty$$

for constants b and B.

Proof. Set $u_{min}(t) = \inf_{y \in M} u(y, t)$ and $u_{max}(t) = \sup_{y \in M} u(y, t)$. By (5.3):

$$\frac{\partial u}{\partial t} = u^{-\frac{2}{n}} \triangle_P^0 u - \frac{1}{\mu} W_0 u^{1-\frac{2}{n}} + \frac{1}{\mu}\overline{W} u,$$

we have

$$\frac{du_{min}}{dt} \geq -\frac{1}{\mu} W_0 u_{min}^{1-\frac{2}{n}} + \frac{1}{\mu}\overline{W} u_{min}.$$

Since

$$\overline{W} = \frac{\int_M W dv}{\int_M dv} = \frac{\int_M (\mu |du|_P^2 + W_0 u^2) dv_0}{\int_M u^p dv_0} \geq -\alpha$$

for some $\alpha > 0$, it follows that

$$u_{min}(t) \geq \min\{u_{min}(0), (\alpha^{-1} \min |W_0|)^{\frac{n}{2}}\}.$$

In a similar way,

$$\frac{du_{max}}{dt} \leq -\frac{1}{\mu} W_0 u_{max}^{1-\frac{2}{n}} + \frac{1}{\mu} \overline{W} u_{max}.$$

Since $\overline{W} \leq \overline{W}(0) < 0$, it implies

$$u_{max} \leq \max\{u_{max}(0), (\max |W_0| |\overline{W}(0)|^{-1})^{\frac{n}{2}}\}.$$

■

In the proof of the short time existence, we get (5.12) to give a bound of the gradient on some short interval. However, estimate (5.12) is not enough to give a bound for $|\nabla u|_{g_{\theta_0}}$ on the interval $[0, T^*)$. The way to improve estimate (5.12) relies on the following lemma.

Setting $W_{max}(t) = \sup_{y \in M} W(y, t)$ and $W_{min}(t) = \inf_{y \in M} W(y, t)$.

Lemma 5.5 *Assume $W(0) < 0$, then*

$$-\delta_1 = W_{min}(0) \leq W \leq W_{max}(0) = -\delta_2 < 0. \tag{5.14}$$

Since $W = (-\mu \triangle_P^0 u + W_0 u) u^{1-p}$, it follows that $|\triangle_P^0 u|$ is uniformly bounded.

Proof. Assume at (x_1, t) and (x_2, t), $W_{max}(t)$ and $W_{min}(t)$ are obtained respectively. It follows from the evolution equation (5.6),

$$\frac{\partial}{\partial t} W(x_1, t) = \frac{n\mu}{2} \triangle_P^\theta W(x_1, t) + W_{max}(W_{max} - \overline{W}) \leq 0$$

and

$$\frac{\partial}{\partial t} W(x_2, t) = \frac{n\mu}{2} \triangle_P^\theta W(x_2, t) + W_{min}(W_{min} - \overline{W}) \geq 0.$$

So W_{max} is nonincreasing and W_{min} is nondecreasing. Thus we have

$$W_{min}(0) \leq W \leq W_{max}(0).$$

■

Proof of the long-time existence. The evolution equation of u is (5.3):

$$\frac{\partial u}{\partial t} = u^{-\frac{2}{n}} (\triangle_P^0 u - \frac{1}{\mu} W_0 u + \frac{1}{\mu} \overline{W} u^{1+\frac{2}{n}}).$$

Assume at $x\in M$, $|\nabla u|^2(x,t)=\sup_{y\in M}|\nabla u|^2(y,t)$. Thus, by lemma 5.5 we have

$$
\begin{aligned}
&\frac{\partial}{\partial t}|\nabla u|^2(x,t)=2\nabla_k u\nabla_k(\frac{\partial u}{\partial t})\\
=\ & 2\nabla_k u\nabla_k(u^{-\frac{2}{n}})(\triangle_P^0 u-\frac{1}{\mu}W_0u+\frac{1}{\mu}\overline{W}u^{1+\frac{2}{n}})\\
&+2\nabla_k uu^{-\frac{2}{n}}\nabla_k(\triangle_P^0 u-\frac{1}{\mu}W_0u+\frac{1}{\mu}\overline{W}u^{1+\frac{2}{n}})\\
=\ & -\frac{4}{n}u^{-1-\frac{2}{n}}|\nabla u|^2(\gamma^{ij}\nabla_i\nabla_j u-\frac{1}{\mu}W_0u+\frac{1}{\mu}\overline{W}u^{1+\frac{2}{n}})\\
&+2\nabla_k uu^{-\frac{2}{n}}\nabla_k(\gamma^{ij}\nabla_i\nabla_j u-\frac{1}{\mu}W_0u+\frac{1}{\mu}\overline{W}u^{1+\frac{2}{n}})\\
\leq\ & c_1+c_2|\nabla u|^2+2u^{-\frac{2}{n}}\nabla_k u\nabla_k\gamma^{ij}\nabla_i\nabla_j u\\
&+2u^{-\frac{2}{n}}(\frac{1}{2}\triangle_P^0|\nabla u|^2-\gamma^{ij}\nabla_i\nabla_k u\nabla_j\nabla_k u-\gamma^{ij}R_{kilj}\nabla_k u\nabla_l u)\\
\leq\ & c_1+c_2|\nabla u|^2(x,t),
\end{aligned}
$$

where we have used the assumption of γ^{ij} as before, so term $2u^{-\frac{2}{n}}\nabla_k u\nabla_k\gamma^{ij}\nabla_i\nabla_j u$ can be dominated by $-2u^{-\frac{2}{n}}\gamma^{ij}\nabla_i\nabla_k u\nabla_j\nabla_k u$.

Therefore $|\nabla u|_{g_{\theta_0}}$ can increase at exponential rate at most. This is enough to give $|\nabla u|_{g_{\theta_0}}$ a bound on interval $[0,T^*)$ for $0<T^*<\infty$.

Almost the same argument can be applied to estimate $|\nabla^2 u|$. Assume at $x\in M$, $|\nabla^2 u|(x,t)=\sup_{y\in M}|\nabla^2 u|(y,t)$. Because u and $|\nabla u|$ are bounded, it follows:

$$
\begin{aligned}
&\frac{\partial}{\partial t}|\nabla^2 u|^2(x,t)=2\nabla_k\nabla_l u\nabla_k\nabla_l(\frac{\partial u}{\partial t})\\
=\ & 2\nabla_k\nabla_l u\nabla_k\nabla_l[u^{-\frac{2}{n}}(\gamma^{ij}\nabla_i\nabla_j u-\frac{1}{\mu}W_0u+\frac{1}{\mu}\overline{W}u^{1+\frac{2}{n}})]\\
=\ & 2\nabla_k\nabla_l u\nabla_k\nabla_l(u^{-\frac{2}{n}})(\gamma^{ij}\nabla_i\nabla_j u-\frac{1}{\mu}W_0u+\frac{1}{\mu}\overline{W}u^{1+\frac{2}{n}})\\
&+4\nabla_k\nabla_l u\nabla_k(u^{-\frac{2}{n}})\nabla_l(\gamma^{ij}\nabla_i\nabla_j u-\frac{1}{\mu}W_0u+\frac{1}{\mu}\overline{W}u^{1+\frac{2}{n}})\\
&+2u^{-\frac{2}{n}}\nabla_k\nabla_l u\nabla_k\nabla_l(\gamma^{ij}\nabla_i\nabla_j u-\frac{1}{\mu}W_0u+\frac{1}{\mu}\overline{W}u^{1+\frac{2}{n}})\\
\leq\ & c_1+c_2|\nabla^2 u|^2+c_3|\gamma^{ij}\nabla_k u\nabla_k\nabla_l u\nabla_l\nabla_i\nabla_j u|\\
&+4u^{-\frac{2}{n}}\nabla_k\nabla_l u\nabla_k\gamma^{ij}\nabla_l\nabla_i\nabla_j u\\
&+2u^{-\frac{2}{n}}\gamma^{ij}\nabla_k\nabla_l u\nabla_k\nabla_l\nabla_i\nabla_j u.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& \frac{\partial}{\partial t}|\nabla^2 u|^2(x,t) \\
\leq\ & c_1 + c_2|\nabla^2 u|^2 + c_3|\gamma^{ij}\nabla_k u \nabla_k \nabla_l u \nabla_l \nabla_i \nabla_j u| \\
& +4u^{-\frac{2}{n}}\nabla_k\nabla_l u \nabla_k \gamma^{ij}\nabla_l\nabla_i\nabla_j u \\
& +2u^{-\frac{2}{n}}\nabla_k\nabla_l u \gamma^{ij}\nabla_i\nabla_j\nabla_k\nabla_l u \\
\leq\ & c_1 + c_2|\nabla^2 u|^2 + c_3|\gamma^{ij}\nabla_k u \nabla_k \nabla_l u \nabla_i \nabla_j \nabla_l u| \\
& +4u^{-\frac{2}{n}}\nabla_k\nabla_l u \nabla_k \gamma^{ij}\nabla_i\nabla_j\nabla_l u \\
& +u^{-\frac{2}{n}}\triangle_P^0(|\nabla^2 u|^2) - 2u^{-\frac{2}{n}}\gamma^{ij}\nabla_i\nabla_k\nabla_l u \nabla_j \nabla_k \nabla_l u \\
\leq\ & c_1 + c_2|\nabla^2 u|^2(x,t).
\end{aligned}
$$

Similar computation gives

$$\frac{d}{dt}|\nabla^k u|^2_{max} \leq c_1 + c_2|\nabla^k u|^2_{max}$$

for $k \geq 3$. With these norm estimates we have finished the proof of the long time existence. ∎

5.2.2 The asymptotic behavior

In this section we prove flow (5.1) converges under the assumption $W(0) < 0$. We proceed by proving quantity $f := (\frac{W}{\overline{W}} - 1)^2$ tend to 0 exponentially as $t \to \infty$, thus the limit θ_∞ has constant Webster scalar curvature if it's smooth. We compute the evolution equation of f at first. Set $h = \frac{W}{\overline{W}} - 1$. By (5.5) and (5.6), it follows that

$$
\begin{aligned}
\frac{\partial h}{\partial t} &= \frac{\partial}{\partial t}(\frac{W}{\overline{W}} - 1) = \frac{1}{\overline{W}}\frac{\partial W}{\partial t} - \frac{W}{\overline{W}^2}\frac{\partial \overline{W}}{\partial t} \\
&= \frac{n\mu}{2}\triangle_P^\theta(\frac{W}{\overline{W}} - 1) + W(\frac{W}{\overline{W}} - 1) + n\frac{W}{\overline{W}^2}\int_M (\overline{W} - W)^2 dv \\
&= \frac{n\mu}{2}\triangle_P^\theta h + Wh + nW\int_M h^2 dv
\end{aligned} \tag{5.15}
$$

and

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= 2(\frac{W}{\overline{W}} - 1)\frac{\partial}{\partial t}(\frac{W}{\overline{W}} - 1) \\
&= \frac{n\mu}{2}\triangle_P^\theta f - n\mu|dh|_P^2 + 2Wf + 2nW(\frac{W}{\overline{W}} - 1)\int_M f dv.
\end{aligned}
$$

The last term $2nW(\frac{W}{\overline{W}} - 1)\int_M f dv$ is probably a bad term which makes the maximum principle invalid when $(\frac{W}{\overline{W}} - 1) < 0$. To resolve this problem, we use term $2Wf$ to dominate the bad term. This will be possible if we can show $|\frac{W}{\overline{W}} - 1|$ is small enough after some time T.

Lemma 5.6 *For any $\epsilon > 0$, there exists $T > 0$ such that $|\frac{W}{\overline{W}} - 1| < \epsilon$ for $t \geq T$.*

Proof. Setting $h_{max}(t) = \sup_{y \in M} h(y,t)$. Formula (5.15) implies

$$\frac{d}{dt} h_{max}(t) \leq -\delta_2 h_{max},$$

since $W \leq -\delta_2$. It follows

$$h \leq ce^{-\delta_2 t}. \tag{5.16}$$

By (5.6), we have

$$\frac{d}{dt} W_{max}(t) = \frac{n\mu}{2} \triangle_P^\theta W + W_{max}(W_{max} - \overline{W}),$$

together with the fact that W is bounded by $-\delta_1 \leq W \leq -\delta_2$, we see $(W_{max} - \overline{W})(t_i)$ has to tend to 0 for some sequence $t_i \to \infty$. Together with (5.16), there exists some $T > 0$ such that $|h(T)| \leq \epsilon$. Therefore,

$$\frac{d}{dt} f_{max}(T) \leq 2W f_{max} + \epsilon \int_M f dv \leq 0$$

for sufficiently small ϵ. It follows from $f = h^2$ that $|h(t)| \leq \epsilon$ for $t \geq T$. ∎

Theorem 5.7 *Along the contact Yamabe flow (5.1) with $W(0) < 0$, f converges to 0 at exponential rate. Thus W approaches $\overline{W}$ exponentially.*

Proof. Since $|\frac{W}{\overline{W}} - 1| \leq \epsilon$ for all $t \geq T$, we have

$$\frac{d}{dt} f_{max} \leq W f_{max} \leq -\delta_2 f_{max}.$$

It follows that

$$f(x,t) \leq ce^{-\delta_2 t}.$$

∎

As a corollary, we have

Theorem 5.8 *Along the contact Yamabe flow (5.1) with $W(0) < 0$, u converges to a continuous limit $u(\infty)$ in C^0 norm.*

Proof. This is because we have $\frac{\partial u}{\partial t} = \frac{n}{2}(\overline{W} - W)u$, so we can compute that

$$\int_T^\infty |\frac{\partial u}{\partial t}| \leq ce^{-\delta_2 T}.$$

∎

5.2.3 Regularity of the limit solution

In this section we prove the limit solution we got in last section is in fact smooth. We need a more general analytic theory than the theorems of Folland and Stein [FS74] which are built on CR manifolds.

Let L be the differential operator given by

$$L = \sum_{j=1}^{m} X_j^2 + X_0,$$

where $X_0, X_1, \cdots, X_m$ are real smooth vector fields on M. Let $X_{j_1} \cdots X_{j_l}$ be a monomial with $0 \le j_s \le m, s = 1, \cdots, l$. We shall say that this monomial has weight r if $r = r_1 + 2r_2$, where r_2 is the number of $X_j's$ that enter with j between 1 and m, and r_2 is the number of $X_0's$. So in computing the total weight, we count each $X_1, X_2, \cdots, X_m$ of weight 1 and X_0 of weight 2. Similarly, the weight of a commutator $[X_{j_1}[X_{j_2}[\cdots, X_{j_l}]\cdots]]$ is defined to be the weight of the corresponding monomial $X_{j_1}X_{j_2}\cdots X_{j_l}$.

Let $k \ge 0$ be an integer, S_k^q is the collection of all $f \in L^q(M)$ such that $X_{j_1}X_{j_2}\cdots X_{j_l}f \in L^q(M)$ for all monomials of weight $\le k$. For the norm we take

$$||f||_{S_k^q} = \sum ||X_{j_1}X_{j_2}\cdots X_{j_l}f||_{L^q(M)},$$

where the sum is taken over all ordered monomials of weight $\le k$. We say that the system of vector fields $\{X_1, \cdots, X_m, X_0\}$ satisfies the Hörmander condition of weight r if these vector fields, together with their commutators of weight r, span the tangent space at any point.

The subelliptic operator $\triangle_P$ defined on any contact metric manifold satisfies the *Hörmander* condition of weight at most $r = 2$. In fact, take $\{X_i \in ker\theta_0\}_{i=1}^{2n}$ as an orthonormal frame with $X_{n+i} = JX_i$ for i between 1 and n, then

$$\triangle_P = \triangle - \xi^2 = X_i^2 - \nabla_{X_i}X_i.$$

Since

$$1 = d\theta_0(X_i, JX_i) = -\theta_0([X_i, JX_i]),$$

the Reeb vector field is generated by the commutator $[X_i, JX_i]$.

The main regularity result for the solution of $L(f) = g$ is the following, due to Rothschild and Stein [RS76].

Theorem 5.9 *Suppose $L = \sum_{j=1}^{m} X_j^2 + X_0$, where all commutators of weight $\le r$ span the tangent space at each point, and $L(f) = g$, $f \in L^p(M)$, $1 < p < \infty$. Then*

(a) If $g \in L_\alpha^q(M)$, then $f \in L_{\alpha+(2/r)}^q(M)$, $\alpha \ge 0$;

(b) If $g \in \Lambda_\alpha(M)$, *then* $f \in \Lambda_{\alpha+(2/r)}(M)$, $\alpha > 0$;

(c) If $g \in L^\infty(M)$, *then* $f \in \Lambda_{2/r}$;

(d) If $g \in S_k^q(M)$, *then* $af \in S_{k+2}^q(M)$, *for each* $a \in C_0^\infty(M)$,

where $1 < q < \infty, k \geq 0$, $L_\alpha^q(M)$ *is the fractional Sobolev space and* Λ_α *is the standard Hölder space.*

By (5.2), we have

$$\mu \triangle_P^0 u = W_0 u - W u^{p-1}.$$

Therefore by Theorem 5.9(a) and 5.9(c), one can see for any $q \geq 2$ $||u||_{S_1^q \cap \Lambda_1}$ is uniformly bounded. Thus there exists a time sequence $\{t_i\}$ such that $u(t_i)$ converges to a limit $u(\infty)$ in $S_1^2 \cap C^0$ which satisfies

$$-\mu \triangle_P^0 u_\infty + W_0 u_\infty = \overline{W}_\infty u_\infty^{p-1} \tag{5.17}$$

in the weak sense.

We can write equation (5.17) as $Lu_\infty = g$, with $L = \mu \triangle_P^0$ and $g = W_0 u_\infty - \overline{W}_\infty u_\infty^{p-1}$. Since $g \in L^\infty(M)$, by induction on Theorem 5.9(a) we see u_∞ is smooth.

5.3 The contact Yamabe flow on K-contact manifolds

In this section we assume the contact metric manifold (M, θ_0, J, g_0) is K-contact, i.e. $L_{\xi_0} g_0 = 0$, where L denotes the Lie derivative and ξ_0 is the Reeb vector field. We show that the contact Yamabe flow (5.1) with initial data $\theta(0) = \theta_0$ has long-time existence and smooth convergence.

5.3.1 Basic material on K-contact manifolds

At first we characterize the K-contact structure (M, θ, J, g). Most of the material is well known. We refer the reader to [Bla02] and [YK84].

Definition 5.10 *Let* (M, θ, J, g) *be a contact metric manifold. If the Reeb vector field* ξ *is a Killing vector field w.r.t.* g, *then* (M, θ, J, g) *is called a* **K-contact (metric) manifold**.

Proposition 5.11 (M, θ, J, g) *is K-contact if and only if* $L_\xi J = 0$.

Proof. Since $(L_\xi d\theta)(X, Y) = 0$ and $g(JX, Y) = d\theta(X, Y)$,

$$\begin{aligned} 0 &= \xi g(JX, Y) - g(J[\xi, X], Y) - g(JX, [\xi, Y]) \\ &= (L_\xi g)(JX, Y) + g([\xi, JX], Y) - g(J[\xi, X], Y) \\ &= (L_\xi g)(JX, Y) + g((L_\xi J)X, Y). \end{aligned}$$

Thus, ξ is a Killing vector field if and only if $L_\xi J = 0$. ∎

We now describe integrable CR structures and K-contact structures. They are both related to vanishing condition of a torsion.

Let (M, θ, J, g) be a contact metric manifold. Consider a product manifold $M \times \mathbb{R}$, where $\mathbb{R}$ denotes a real line. Then a vector field on $M \times \mathbb{R}$ is given by $(X, f\frac{\partial}{\partial t})$, where X is a vector field tangent to M, t the coordinate of $\mathbb{R}$ and f a function on $M \times \mathbb{R}$. Extend J to $\overline{J}$ defined on the tangent space of $M \times \mathbb{R}$ by

$$\overline{J}(X, f\frac{\partial}{\partial t}) = (JX - f\xi, \theta(X)\frac{\partial}{\partial t}).$$

Then $\overline{J}^2 = -I$ and hence $\overline{J}$ is an almost complex structure on $M \times \mathbb{R}$. The almost complex structure $\overline{J}$ is said to be integrable if its Nijenhuis torsion $N_{\overline{J}}$ vanishes, where

$$N_{\overline{J}}(X, Y) = \overline{J}^2[X, Y] + [\overline{J}X, \overline{J}Y] - \overline{J}[\overline{J}X, Y] - \overline{J}[X, \overline{J}Y].$$

If the almost complex structure $\overline{J}$ is integrable, we say that the contact metric manifold (M, θ, J, g) is **normal**.

Denote

$$N(X, Y) = N_J(X, Y) + d\theta(X, Y)\xi \quad for \quad X, Y \in TM.$$

(M, θ, J, g) is normal if and only if $N = 0$ (see [YK84]). If the contact metric manifold (M, θ, J, g) is normal, then M is said to have a **Sasakian structure** (or **normal contact metric structure**) and (M, θ, J, g) is called a **Sasakian manifold** (or **normal contact metric manifold**). So a Sasakian manifold is sometimes viewed as an odd dimensional analogue of a *Kähler* manifold.

Note that a CR manifold (M, θ, J, g) is a contact metric manifold with integrable J in the sense:

$$N_J(X, Y) + d\theta(X, Y)\xi = 0 \quad for \quad X, Y \in G = ker\theta.$$

Therefore a Sasakian manifold is CR manifold containing more data. The remaining data is just $N(X, \xi) = 0 \quad for \quad X \in G$.

Proposition 5.12 *$N(X, \xi) = 0 \quad for \quad X \in G$ if and only if M is K-contact.*

Proof.

$$\begin{aligned} 0 &= N(JX, \xi) \\ &= J^2[JX, \xi] + J[X, \xi] \\ &= [\xi, JX] + J[X, \xi] \\ &= (L_\xi J)(X). \end{aligned}$$

Therefore by proposition 5.11, we finish the proof. ■

Now it's clear that a contact metric manifold (M, θ, J, g) is Sasakian if and only if it's CR and K-contact.

In the following we collect some necessary facts needed later.

Proposition 5.13 *(M, θ, J, g) is K-contact if and only if $\nabla_X \xi = \frac{1}{2} JX$.*

Proof. $(L_\xi g)(X, Y) = 0$ implies

$$\begin{aligned} 0 &= \xi g(X, Y) - g([\xi, X], Y) - g(X, [\xi, Y]) \\ &= g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi). \end{aligned}$$

Therefore,

$$\begin{aligned} g(JX, Y) &= d\theta(X, Y) \\ &= X\theta(Y) - Y\theta(X) - \theta([X, Y]) \\ &= Xg(Y, \xi) - Yg(X, \xi) - g([X, Y], \xi) \\ &= g(Y, \nabla_X \xi) - g(X, \nabla_Y \xi) \\ &= 2g(Y, \nabla_X \xi). \end{aligned}$$

So we have $\nabla_X \xi = \frac{1}{2} JX$. Conversely, if $\nabla_X \xi = \frac{1}{2} JX$,

$$\begin{aligned} (L_\xi g)(X, Y) &= g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \\ &= g(\frac{1}{2} JX, Y) + g(X, \frac{1}{2} JY) \\ &= 0. \end{aligned}$$

■

Proposition 5.14 *(M, θ, J, g) is K-contact if and only if $\nabla_i \theta_j + \nabla_j \theta_i = 0$.*

Proof. $(L_\xi g)(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = 0$ implies

$$g(\nabla_i \xi, \frac{\partial}{\partial x^j}) + g(\frac{\partial}{\partial x^i}, \nabla_j \xi) = 0.$$

Therefore, $\nabla_i \theta_j + \nabla_j \theta_i = 0$. ■

Lemma 5.15 *Let (M, θ, J, g) be a K-contact manifold. we have*

$$\nabla_i J^i_j = n\theta_j.$$

Proof. Denote

$$\begin{aligned} \phi_{ij} &= d\theta(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \\ &= \nabla_i \theta_j - \nabla_j \theta_i \\ &= J^k_i g_{jk}. \end{aligned}$$

Therefore by proposition 5.13, we have

$$\nabla_r J_j^r \xi^j = -\nabla_r \xi^j J_j^r = -\nabla^s \xi^j \phi_{js} = -\frac{1}{2} J_r^j J_j^r = n.$$

So we have

$$(\nabla_r J_j^r - n\theta_j)\xi^j = 0.$$

On the other hand,

$$(\nabla_r J_j^r - n\theta_j) J_k^j = -\nabla_r \phi_{kl} g^{jl} J_j^r = 0,$$

by using the closeness of $d\theta$. So we have $\nabla_i J_j^i = n\theta_j$. ∎

Proposition 5.16 *([Bla02]) For K-contact metric manifold (M, θ, J, g), we have*

$$Ric(\xi, \frac{\partial}{\partial x^j}) = \frac{1}{2} n\theta_j.$$

Proof. In normal coordinates, we have

$$\begin{aligned}
Ric(\xi, \frac{\partial}{\partial x^j}) &= R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}, \xi) \\
&= < \nabla_i \nabla_j \xi - \nabla_j \nabla_i \xi, \frac{\partial}{\partial x^i} > \\
&= \frac{1}{2} < \nabla_i (J_j^k \frac{\partial}{\partial x^k}) - \nabla_j (J_i^k \frac{\partial}{\partial x^k}), \frac{\partial}{\partial x^i} > \\
&= \frac{1}{2} (\nabla_i J_j^k - \nabla_j J_i^k) g_{ki} \\
&= \frac{1}{2} \nabla_i J_j^i \\
&= \frac{1}{2} n\theta_j.
\end{aligned}$$

∎

Therefore we see $Ric(\xi, \xi) = \frac{1}{2} n$. A theorem of Blair [Bla02] says this is also a sufficient condition.

The second Bianchi identity implies

$$\xi R = 2\xi^i \nabla^j R_{ij}.$$

Tanno [Tan89] got the relation between the Webster scalar curvature W and the scalar curvature R, i.e. formula (3.9):

$$W = R - Ric(\xi, \xi) + 4n.$$

Proposition 5.17 *For a K-contact metric manifold (M, θ, J, g), we have*

$$\xi W = 0.$$

Proof. Thanks to proposition 5.16, we have

$$Ric(\xi,\xi)=\frac{1}{2}n,$$

and

$$\begin{aligned}\xi R &= 2\xi^i\nabla^j R_{ij}=2\nabla^j(Ric(\xi,\frac{\partial}{\partial x^j}))-2R_{ij}\nabla^j\xi^i\\ &= 2\nabla^j(\frac{1}{2}n\theta_j)-R_{ij}(\nabla^j\xi^i+\nabla^i\xi^j)\\ &= 0.\end{aligned}$$

■

5.3.2 The long-time existence and convergence

In this section we consider the contact Yamabe flow on a K-contact metric manifold (M,θ_0,J,g_0) with initial data $\theta(0)=\theta_0$, i.e.

$$\begin{cases}\frac{\partial\theta}{\partial t}=(\overline{W}-W)\theta\\ \theta(0)=\theta_0\end{cases}$$

We say a function u is **basic** if $\xi u=0$. In particular, $u(0)=1$ is basic.

Lemma 5.18 *On a K-contact metric manifold (M,θ_0,J,g_0), we have $\xi\triangle^0 f=\triangle^0\xi f$ for any $f\in C^\infty(M)$.*

Proof. Choose normal coordinates, we compute

$$\begin{aligned}\nabla_i\nabla_i(\xi^k\nabla_k f) &= \xi^k\nabla_i\nabla_i\nabla_k f+2\nabla_i\xi^k\nabla_i\nabla_k f+\nabla_i\nabla_i\xi^k\nabla_k f\\ &= \xi^k\nabla_i\nabla_i\nabla_k f+\nabla_i\nabla_i\xi^k\nabla_k f\\ &= \xi^k\nabla_i\nabla_i\nabla_k f-\nabla_i\nabla_k\theta_i\nabla_k f\\ &= (\xi^k\nabla_k\nabla_i\nabla_i f-\xi^k R_{ikli}\nabla_l f)-(\nabla_k\nabla_i\theta_i\nabla_k f-R_{ikli}\theta_l\nabla_k f)\\ &= \xi\triangle^0 f+Ric(\xi,\nabla f)-Ric(\xi,\nabla f)\\ &= \xi\triangle^0 f,\end{aligned}$$

where we have used the facts $div(\xi)=0$ and $\nabla_i\theta_j+\nabla_j\theta_i=0$.

Theorem 5.19 *Assume (M,θ_0,J,g_0) is K-contact. Then along the contact Yamabe flow (5.1) with $\theta(0)=\theta_0$ u is basic.*

Proof. Since $\xi W_0=0$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}|\xi u|^2=\xi u\xi(\frac{\partial u}{\partial t})\\ =\ &\xi u\xi(u^{-\frac{2}{n}}\triangle^0_P u-\frac{1}{\mu}W_0u^{1-\frac{2}{n}}+\frac{1}{\mu}\overline{W}u)\\ =\ &u^{-\frac{2}{n}}\xi u\xi(\triangle^0_P u)-\frac{2}{n}u^{-1-\frac{2}{n}}\triangle^0_P u|\xi u|^2-\frac{1}{\mu}(1-\frac{2}{n})u^{-\frac{2}{n}}W_0|\xi u|^2+\frac{1}{\mu}\overline{W}|\xi u|^2.\end{aligned}$$

Then by lemma 5.18, we have

$$
\begin{aligned}
& \frac{1}{2}\frac{\partial}{\partial t}|\xi u|^2 \\
= \;& u^{-\frac{2}{n}}\xi u \triangle_P^0 \xi u - \frac{2}{n}u^{-1-\frac{2}{n}}\triangle_P^0 u|\xi u|^2 - \frac{1}{\mu}(1-\frac{2}{n})W_0 u^{-\frac{2}{n}}|\xi u|^2 + \frac{1}{\mu}\overline{W}|\xi u|^2 \\
= \;& \frac{1}{2}u^{-\frac{2}{n}}\triangle_P^0|\xi u|^2 - u^{-\frac{2}{n}}|d\xi u|_P^2 \\
& +(-\frac{2}{n}u^{-1-\frac{2}{n}}\triangle_P^0 u - \frac{1}{\mu}(1-\frac{2}{n})W_0 u^{-\frac{2}{n}} + \frac{1}{\mu}\overline{W})|\xi u|^2.
\end{aligned}
$$

It follows that $\xi u = 0$ whenever the solution exists. ■

Take an orthonormal frame $\{\xi, X_i : i = 1, \cdots, 2n\}$ with respect to (M, θ_0, J, g_0). Therefore $\triangle_P^0 = \triangle^0 - \xi^2 = X_i^2 - \nabla_{X_i}X_i$. Since $g_0(\xi, X_i) = 0$, $X_i \in ker\theta_0$. Denote $X_0 = -\nabla_{X_i}X_i$.

Set $v = \log u$. The evolution equation of v is

$$
\frac{\partial v}{\partial t} = e^{-\frac{2}{n}v}(\sum_i X_i^2 v + \sum_i |X_i v|^2 + X_0 v - \frac{1}{\mu}W_0) + \frac{1}{\mu}\overline{W}.
$$

We compute

$$
\begin{aligned}
& \frac{1}{2}\frac{\partial}{\partial t}(\sum_k |X_k v|^2) = X_k v X_k(\frac{\partial v}{\partial t}) \\
= \;& -\frac{2}{n}e^{-\frac{2}{n}v}(\sum_k |X_k v|^2)(\sum_i X_i^2 v + \sum_i |X_i v|^2 + X_0 v - \frac{1}{\mu}W_0) \\
& +e^{-\frac{2}{n}v}(X_k v X_k X_i^2 v + X_k v X_k |X_i v|^2 + X_k v X_k X_0 v - \frac{1}{\mu}X_k W_0 X_k v).
\end{aligned}
$$

It follows from

$$
X_k X_i^2 - X_i^2 X_k = [X_k, X_i]X_i + X_i[X_k, X_i],
$$

$$
\begin{aligned}
& \frac{1}{2}\frac{\partial}{\partial t}(\sum_k |X_k v|^2) \\
= \;& e^{-\frac{2}{n}v}\{\frac{1}{2}X_i^2|X_k v|^2 - |X_i X_k v|^2 - \frac{2}{n}(\sum_k |X_k v|^2)^2 - \frac{2}{n}|X_k v|^2 X_i^2 v \\
& +X_k v X_k|X_i v|^2 - \frac{2}{n}|X_k v|^2 X_0 v + \frac{2}{n\mu}W_0|X_k v|^2 + X_k v X_k X_0 v \\
& -\frac{1}{\mu}X_k W_0 X_k v + X_k v[X_k, X_i]X_i v + X_k v X_i[X_k, X_i]v\}.
\end{aligned}
$$

Assume at time t, $|\nabla v|$ attains its maximum at point x_0, here

$$|\nabla v| = (\sum_k |X_k v|^2)^{\frac{1}{2}}$$

is the gradient with respect to (M, θ_0, J, g_0). To simplify the computation, we choose $\{X_i\}_{i=1}^{2n}$ so that $X_1 v(x_0, t) = |\nabla v|, X_k v = 0$ for any $k \neq 1$. We have used the fact $\xi v = 0$.

Since $|\nabla v| \geq X_1 v$ and $|\nabla v|(x_0, t) = X_1 v(x_0, t)$, it follows that

$$X_1^2 v(x_0, t) = X_1 |\nabla v|(x_0, t) = 0.$$

Denote the indices from 2 to 2n by α. We now have

$$\begin{aligned} & -|X_\alpha^2 v|^2 - \frac{2}{n}|X_1 v|^4 - \frac{2}{n}|X_1 v|^2 X_\alpha^2 v \\ \leq \;& \sum_\alpha (-|X_\alpha^2 v|^2 - \frac{2}{n(2n-1)}|X_1 v|^4 + \frac{2}{n}|X_1 v|^2 |X_\alpha^2 v|). \end{aligned}$$

Fortunately, $(\frac{2}{n})^2 < 4(\frac{2}{n(2n-1)})$, we have

$$\frac{1}{2}\frac{\partial}{\partial t}(\sum_k |X_k v|^2)(x_0, t) \leq e^{-\frac{2}{n}v}(-c_1(\sum_k |X_k v|^2)^2 + c_2 \sum_k |X_k v|^2 + c_3),$$

for positive constants c_1, c_2 and c_3. It implies

$$\sum_k |X_k v|^2 \leq c.$$

We have proved the following.

Theorem 5.20 *Assume (M, θ_0, J, g_0) is K-contact. Then along the contact Yamabe flow (5.1) with initial data $\theta(0) = \theta_0$, $|\nabla v| \leq c$, i.e. the gradient is uniformly bounded.*

Corollary 5.21 *Assume (M, θ_0, J, g_0) is K-contact. Then along the contact Yamabe flow (5.1), $0 < \delta \leq u \leq c < \infty$, and $|\nabla u| \leq c$.*

Proof. Since for any two points x and y

$$|\log \frac{u(x)}{u(y)}| = |v(x) - v(y)| \leq c|\nabla v| \leq c,$$

it implies

$$u(x)/u(y) \leq c.$$

It follows from $\int_M u^{2+\frac{2}{n}} = 1$ that $0 < \delta \leq u \leq c < \infty$. Therefore, $|\nabla v| = \frac{|\nabla u|}{u} \leq c$ implies $|\nabla u| \leq c$. ∎

One can observe that u is in fact the solution of
$$\frac{\partial u}{\partial t} = u^{-\frac{2}{n}}\triangle^0 u - \frac{1}{\mu}W_0 u^{1-\frac{2}{n}} + \frac{1}{\mu}\overline{W}u,$$
since
$$\frac{\partial u}{\partial t} = u^{-\frac{2}{n}}\triangle^0_P u - \frac{1}{\mu}W_0 u^{1-\frac{2}{n}} + \frac{1}{\mu}\overline{W}u + u^{-\frac{2}{n}}\xi^2 u$$
and the solution of
$$\frac{\partial u}{\partial t} = u^{-\frac{2}{n}}\triangle^0 u - \frac{1}{\mu}W_0 u^{1-\frac{2}{n}} + \frac{1}{\mu}\overline{W}u$$
is unique. So we have the following.

Theorem 5.22 *If (M,θ_0,J,g_0) is K-contact, along the contact Yamabe flow (5.1) with $\theta(0)=\theta_0$ all derivatives of u are uniformly bounded therefore flow (5.1) has long-time existence.*

Denote $f(t)=\int_M (W(t)-\overline{W}(t))^2 dv$. Since all derivatives of u are bounded, $|\frac{df}{dt}|$ is also uniformly bounded. On the other hand since
$$\frac{d}{dt}\int_M W dv = -n\int_M (W-\overline{W})^2 dv,$$
it follows that
$$\int_0^\infty f(t) \le c.$$
Therefore $f(t)$ tend to 0 uniformly.

Since $|\nabla W|$ is bounded and $f(t)$ tends to 0 uniformly, we see W converges to some constant smoothly. So we have the following.

Theorem 5.23 *If (M,θ_0,J,g_0) is K-contact, the contact Yamabe flow (5.1) with $\theta(0)=\theta_0$ converges smoothly to some limit which has constant Webster scalar curvature.*

As its corollary, we prove

Corollary 5.24 *Let (M,θ_0,J,g_0) is a K-contact metric manifold. Then there exists a conformal contact form $\theta = u^{\frac{2}{n}}\theta_0$ which has constant Webster scalar curvature.*

It's interesting to look at the contact Yamabe flow (5.1) on the standard sphere which admits a standard Sasakian structure. Let (S^{2n+1},θ_1) be the sphere with standard CR structure. We adopt the notations introduced in chapter 4. Let ζ^j be the complex coordinates of $\mathbb{C}^{n+1}$. Then
$$\theta_1 = i\sum_{j=1}^{n+1}(\zeta^j d\overline{\zeta}^j - \overline{\zeta}^j d\zeta^j).$$
The Reeb vector field $\xi = i\sum_{j=1}^{n+1}(\zeta^j\frac{\partial}{\partial\zeta^j} - \overline{\zeta}^j\frac{\partial}{\partial\overline{\zeta}^j})$.

We have seen in section 4.4 that the solutions u on (S^{2n+1}, θ_1) such that $u^{\frac{2}{n}}\theta_1$ has constant Webster scalar curvature are

$$u(\zeta) = K|i(1 - \zeta^{n+1}) + \lambda(1 + \zeta^{n+1}) + \sum_{k=1}^{n} \zeta^k \mu^k|^{-n},$$

where $K \in \mathbb{R}, \lambda \in \mathbb{C}, Im(\lambda) > \frac{|\mu|^2}{4}$ and $\mu \in \mathbb{C}^n$. Denote the set of all solutions by $\mathcal{S}$.

It's a little bit surprising that for K-contact manifold (M, θ_0, J, g_0) along the contact Yamabe flow (5.1) $u(t)$ and all its derivatives are uniformly bounded. However on (S^{2n+1}, θ_1) the total solution set $\mathcal{S}$ is not bounded. Comparably, the contact Yamabe flow on locally flat and scalar positive manifolds, in particular on S^n, is also the case according to [Ye94]. More generally, the results of Schwetlick and Struwe [SS03] , Brendle [Bre05] show along the contact Yamabe flow its solution $u(t)$ is uniformly bounded in the three cases of Theorem 2.29 [Bre05].

The reason is that $u(t)$ break through the total solution set. For example, in our case, the solutions in $\mathcal{S}$ satisfying $\xi u = 0$ are only that of $\mu = 0$ and $\lambda = i$, i.e.

$$u = K|i(1 - \zeta^{n+1}) + i(1 + \zeta^{n+1})|^{-n} = c.$$

Another explanation could be the following. We have observed that along the contact Yamabe flow u is basic. Thus if we turn to look at the contact Yamabe equation, the basic property makes it much like the Riemannian Yamabe equation on a 2n-dimensional manifold. Since $\frac{2(2n)}{(2n)-2} > 2 + \frac{2}{n}$, it suggests that the contact Yamabe flow (5.1) on K-contact manifold is subcritical.

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